Online Supplement to the Paper “Patient Triage and Prioritization under Austere Conditions"

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1 Expressions and derivations for the cost functions

We start with defining the following two policies:

Triage-All-First Policy (TAF): First, all patients go through triage in random order. Then, class-1 patients are given priority in agreement with the $c\mu$-rule, i.e., all class-1 patients are served before all class-2 patients.

Triage-Prioritize-Class-2 Policy (TP2): Each patient, with the exception of the last one, goes through triage in random order. If a patient is classified as class-2, s/he is served right away; otherwise, s/he is put aside to be served later. When the triage of $N-1$ patients is completed, the remaining untriaged patient is served followed by all class-1 patients.

We use V_{NT}, V_{TAF}, V_{TP1} and V_{TP2} to denote the total expected cost under policies *NT, TAF, TP1* and *TP2*. Next, we derive their expressions.

1.1 Expression for V_{NT}

Consider the i^{th} patient to be served, $1 \le i \le N$. The sojourn time in the system is the total service time of the first $i-1$ patients plus its own service time. This leads to

$$V_{NT} = \sum_{i=1}^{N} [c_0 + (i-1)r_0\tau_0] = Nc_0 + \frac{N(N-1)}{2} r_0\tau_0. \tag{1}$$

1.2 Expressions for V_{TP1} and V_{TP2}

We will derive an expression for V_{TP1}. Let N_1 denote the number of class-1 patients among the $N-1$ patients that go through triage. We know $N_1 \sim B(N-1,\alpha_1)$ where $B(n,p)$ indicates a binomial random variable with parameters n and p. Let $N_1 = k$ and $S = \{S_1, S_2, \cdots, S_k\}$ where S_j indicates the order number of the jth class-1 patient when the server is picking among the class-0 patients randomly, i.e. the jth patient classified as class-1 is the S_jth patient to have been picked by the server among the class-0 patients. For $i_1, i_2, \cdots, i_k, 1 \le i_j < N, j = 1, 2, \cdots, k$, we can show that $P\big(S_1 = i_1, S_2 = i_2, \cdots, S_k = i_k \big| N_1 = k\big) = \frac{1}{\binom{N-1}{k}}$.

Conditional on $N_1 = k$ and $S = \{i_1, i_2, \cdots, i_k\}$, the expected cost incurred by k class-1 patients due to the triage of all $N-1$ patients is given by $\Gamma_1\Big|_{S,N_1=k}(triage) = r_1(i_1u+i_2u+\cdots+i_ku) = r_1u\sum_{m=1}^{k} i_m$. Conditional on $N_1 = k$ and $S = \{i_1, i_2, \cdots, i_k\}$, the expected cost incurred by k class-1 patients due to the service of all k class-1 patients is given by $\Gamma_1\Big|_{S,N_1=k}(service) = c_1 + (c_1 + r_1\tau_1) + \cdots + (c_1 + (k-1)r_1\tau_1) = kc_1 + \frac{k(k-1)}{2} r_1\tau_1$. Then, the total expected cost incurred by the k class-1 patients is $\Gamma_1\Big|_{S,N_1=k} = kc_1 + r_1\left(u\sum_{m=1}^{k} i_m + \frac{k(k-1)}{2}\tau_1\right)$.

Define $\mathcal{A}_k = \{$all possible combinations of $(i_1, \cdots, i_k), 1 \leq i_1 < i_2 < \cdots < i_k < N\}$. The total expected cost for the k class-1 patients is

$$
\begin{aligned}
\Gamma_1\Big|_{N_1=k} &= \sum_{\mathcal{A}_k} \Gamma_1\Big|_{S,N_1=k} \cdot P\big(S_1 = i_1, S_2 = i_2, \cdots, S_k = i_k \big| N_1 = k\big) \\
&= \sum_{\mathcal{A}_k} \left[kc_1 + r_1\left(u\sum_{m=1}^{k} i_m + \frac{k(k-1)}{2}\tau_1\right)\right] \cdot P\big(S_1 = i_1, S_2 = i_2, \cdots, S_k = i_k \big| N_1 = k\big) \\
&= kc_1 + \frac{k(k-1)}{2} r_1\tau_1 + r_1 u \sum_{\mathcal{A}_k} \sum_{m=1}^{k} i_m \cdot \frac{1}{\binom{N-1}{k}}.
\end{aligned}
$$

In the term $\sum_{\mathcal{A}_k}\sum_{m=1}^{k} i_m$ each number $i \in \{1, 2, \cdots, N-1\}$ appears exactly $\binom{N-2}{k-1} = k \cdot \binom{N-1}{k}/(N-1)$ times. Hence the total expected cost for class-1 patients is

$$
\begin{aligned}
\Gamma_1\Big|_{N_1=k} &= kc_1 + \frac{k(k-1)}{2} r_1\tau_1 + r_1 u \cdot \frac{k \cdot \binom{N-1}{k}}{N-1} \cdot \frac{N(N-1)}{2} \cdot \frac{1}{\binom{N-1}{k}} \\
&= kc_1 + \frac{k(k-1)}{2} r_1\tau_1 + \frac{Nk}{2} r_1 u. \qquad (2)
\end{aligned}
$$

The expected cost for the last patient, the only patient that does not go through triage, conditional on $N_1 = k$ is

$$
\psi(k) = r_0\left[(N-1)u + k\tau_1\right] + c_0. \qquad (3)
$$

The total expected cost for class-2 patients when there are k class-1 patients is

$$
\begin{aligned}
\Gamma_2\Big|_{N_1=k} &= \Big(r_2[(N-1)u + \tau_0 + k\tau_1] + c_2\Big) + \Big(r_2\left[(N-1)u + \tau_0 + k\tau_1 + \tau_2\right] + c_2\Big) + \cdots \\
&+ \Big(r_2[(N-1)u + \tau_0 + k\tau_1 + (N-k-2)\tau_2] + c_2\Big) \\
&= \sum_{m=k+2}^{N} \Big(c_2 + r_2[(N-1)u + \tau_0 + k\tau_1] + r_2(m-k-2)\tau_2\Big). \qquad (4)
\end{aligned}
$$

Then, the total expected cost under Policy *TP1* is

$$
V_{TP1} = \sum_{k=0}^{N-1}\left[\Gamma_1\Big|_{N_1=k} + \psi(k) + \Gamma_2\Big|_{N_1=k}\right] P(N_1 = k). \qquad (5)
$$

Plugging in (2), (3) and (4) into (5) with some algebraic manipulation we get

$$
\begin{aligned}
V_{TP1} = &\sum_{k=0}^{N-1}\Bigg[\left(kc_1 + \frac{k(k-1)}{2} r_1\tau_1 + \frac{Nk}{2} r_1 u\right) + \Big(r_0\left[(N-1)u + k\tau_1\right] + c_0\Big) \\
&+ \sum_{m=k+2}^{N}\Big(c_2 + r_2[(N-1)u + \tau_0 + k\tau_1] + r_2(m-k-2)\tau_2\Big)\Bigg] P(N_1 = k) \\
= &\ N(N-1)r_0 u - \frac{(N-1)(N-2)}{2}\alpha_1 r_1 u + c_0 + (N-1)(\alpha_1 c_1 + \alpha_2 c_2) \\
&+ (N-1)(\alpha_1 r_0 \tau_1 + \alpha_2 r_2 \tau_0) + (N-1)(N-2)\alpha_1\alpha_2 r_2 \tau_1 \\
&+ \frac{(N-1)(N-2)}{2}(\alpha_1^2 r_1\tau_1 + \alpha_2^2 r_2\tau_2). \qquad (6)
\end{aligned}
$$

We can similarly obtain

$$\begin{aligned} V_{TP2} = {} & N(N-1)r_0u - \frac{(N-1)(N-2)}{2}\alpha_2 r_2 u + c_0 + (N-1)(\alpha_1 c_1 + \alpha_2 c_2) \\ & + (N-1)(\alpha_1 r_1 \tau_0 + \alpha_2 r_0 \tau_2) + (N-1)(N-2)\alpha_1\alpha_2 r_1 \tau_2 \\ & + \frac{(N-1)(N-2)}{2}(\alpha_1^2 r_1 \tau_1 + \alpha_2^2 r_2 \tau_2). \end{aligned} \tag{7}$$

1.3 Expression for V_{TAF}

The derivation of V_{TAF} is similar to that of V_{TP1}. The total expected triage cost for all patients is $Nr_0 \cdot Nu$. Denote N_1 as the number of patients classified as class-1, $N_1 \sim B(N, \alpha_1)$. Then, given $N_1 = k$, the total expected cost incurred by class-1 patients during the service of all class-1 patients is $\Gamma_1\Big|_{N_1=k} = \Big(c_1 + (k-1)r_1\tau_1\Big) + \cdots + \Big(c_1 + (k-k)r_1\tau_1\Big) = kc_1 + \frac{k(k-1)}{2}r_1\tau_1$. Given $N_1 = k$, the total expected cost incurred by class-2 patients during the service of all patients is $\Gamma_2\Big|_{N_1=k} = \Big((N-k)r_2 \cdot k\tau_1\Big) + \Big(c_2 + (N-k-1)r_2\tau_2\Big) + \cdots + \Big(c_2 + 0 \cdot r_2\tau_2\Big) = (N-k)r_2 \cdot k\tau_1 + (N-k)c_2 + \frac{(N-k)(N-k-1)}{2}r_2\tau_2$. Hence, the total expected cost under Policy TAF is

$$\begin{aligned} V_{TAF} &= N^2 r_0 u + \sum_{k=0}^{N}\Big[\Big(kc_1 + \frac{k(k-1)}{2}r_1\tau_1\Big) + \big((N-k)r_2 \cdot k\tau_1 + (N-k)c_2 \\ &\quad + (N-k)(N-k-1)r_2\tau_2/2\big)\Big]P(N_1 = k) \\ &= N^2 r_0 u + N(\alpha_1 c_1 + \alpha_2 c_2) + N(N-1)\alpha_1\alpha_2 r_2 \tau_1 + \frac{N(N-1)}{2}(\alpha_1^2 r_1 \tau_1 + \alpha_2^2 r_2 \tau_2). \end{aligned} \tag{8}$$

2 Proof of Proposition 1

Part (i): Among all the policies that *perform triage on all the class-0 patients first and serving them later*, we know from the $c\mu$-rule that once the triage of all patients is complete, the optimal action is to prioritize class-1 patients over class-2 patient. Hence, we only need to show $V_{TAF} - V_{TP1} > 0$.

From (6) and (8), we have

$$\begin{aligned} \frac{V_{TAF} - V_{TP1}}{N-1} &= r_0 u + \frac{N-2}{2}\alpha_1 r_1 u + \frac{\alpha_1 c_1 + \alpha_2 c_2 + r_0 u - c_0}{N-1} + 2\alpha_1\alpha_2 r_2 \tau_1 \\ &\quad -(\alpha_1 r_0 \tau_1 + \alpha_2 r_2 \tau_0) + (\alpha_1^2 r_1 \tau_1 + \alpha_2^2 r_2 \tau_2) \\ &= \frac{N}{2}\alpha_1 r_1 u + \frac{\alpha_1 c_1 + \alpha_2 c_2 + r_0 u - c_0}{N-1} + \alpha_2 r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0, \end{aligned}$$

where the last inequality follows from Assumption 1.

Part (ii): With the definition of Policy *TP2*, essentially, we need to show $V_{TP2} - V_{NT} > 0$. From (1) and (7), we have

$$\begin{aligned} \frac{V_{TP2} - V_{NT}}{N-1} &= (\alpha_1 c_1 + \alpha_2 c_2 + r_0 u - c_0) + (N-1)\alpha_1 r_1(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) \\ &\quad + \frac{N}{2}\alpha_2 r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) + \frac{N}{2}\alpha_1\tau_0\tau_1\left(\frac{r_1}{\tau_1} - \frac{r_0}{\tau_0}\right) \\ &= (r_0 u - \tilde{c}) + \left(\frac{N-2}{2}\alpha_1 r_1 + \frac{N}{2}r_0\right)(u - \tilde{\tau}) + \frac{N}{2}\alpha_1\tau_0\tau_1\left(\frac{r_1}{\tau_1} - \frac{r_0}{\tau_0}\right) > 0. \end{aligned}$$

Hence, $V_{TP2} > V_{NT}$. □

3 Proof of Proposition 2

From (1) and (6), we have

$$\begin{aligned}\frac{V_{TP1}-V_{NT}}{N-1} &= (Nr_0-(N-2)\alpha_1 r_1/2)u+(\alpha_1 c_1+\alpha_2 c_2-c_0)+\alpha_1 r_0\tau_1+\alpha_2 r_2\tau_0\\ &\quad +(N-2)\alpha_1\alpha_2 r_2\tau_1+\frac{N-2}{2}(\alpha_1^2 r_1\tau_1+\alpha_2^2 r_2\tau_2)-\frac{N}{2}r_0\tau_0\\ &= \Big(\frac{N}{2}(\alpha_2 r_2+r_0)+\alpha_1 r_1\Big)(u-\tilde{\tau})-(\tilde{c}-r_0\tilde{\tau})-\frac{N}{2}\alpha_2(r_0\tau_2-r_2\tau_0).\end{aligned}$$

When $\frac{r_0}{\tau_0}\geq\frac{r_2}{\tau_2}$, one can then immediately obtain $V_{TP1}-V_{NT}\leq 0$ if and only if $u\leq\beta$ where

$$\beta=\max\left\{\frac{\frac{N}{2}\alpha_2(r_0\tau_2-r_2\tau_0)+\tilde{c}-r_0\tilde{\tau}}{\frac{N}{2}(\alpha_2 r_2+r_0)+\alpha_1 r_1}+\tilde{\tau},0\right\}.$$

□

4 Proof of Theorem 1

We first prove the following lemma:

Lemma 1. *For all* $(i,k_1,k_2)\in\mathcal{S}$*, we have*

(i) $V(i,k_1+1,k_2)\geq V(i,k_1,k_2)+c_1$.

(ii) $V(i,k_1,k_2+1)\geq V(i,k_1,k_2)+c_2$.

(iii) If $u\geq\tau_0$*, then* $a^*(i,k_1,k_2)\neq$ Tr.

Proof of Lemma 1:

Part (i): The proof uses a coupling argument. Consider two systems. System 1 and System 2 are identical except that System 1 starts in state (i,k_1+1,k_2) and uses the optimal policy and System 2 starts in state (i,k_1,k_2) and uses policy π, which takes whatever action System 1 takes until System 1 starts serving the extra class-1 patient System 2 lacks. While System 1 serves the extra class-1 patient, System 2 idles and then follows the same actions as System 1 until all patients are cleared.

Let the total expected cost under policy π be denoted by $V_\pi(i,k_1,k_2)$. The difference between $V(i,k_1+1,k_2)$ and $V_\pi(i,k_1,k_2)$ is at least as large as the expected cost incurred by the additional class-1 patient during its service. Hence,

$$\begin{aligned}V(i,k_1+1,k_2)-V(i,k_1,k_2) &= V(i,k_1+1,k_2)-V_\pi(i,k_1,k_2)+V_\pi(i,k_1,k_2)-V(i,k_1,k_2)\\ &\geq V(i,k_1+1,k_2)-V_\pi(i,k_1,k_2)\geq c_1.\end{aligned}$$

Part (ii): The proof is similar to that for part (i) and is therefore omitted.

Part (iii): In any state (i,k_1,k_2), the total expected cost of first doing *triage* then following the optimal policy is

$$J_{Tr}(i,k_1,k_2)=\alpha_1 V(i-1,k_1+1,k_2)+\alpha_2 V(i-1,k_1,k_2+1)+(ir_0+k_1 r_1+k_2 r_2)u.$$

The total expected cost of first doing SU and then following the optimal policy is

$$J_{SU}(i,k_1,k_2)=V(i-1,k_1,k_2)+c_0+[(i-1)r_0+k_1 r_1+k_2 r_2]\tau_0.$$

By parts (i) and (ii) of the lemma,

$$\begin{aligned} J_{Tr}(i,k_1,k_2) &= \alpha_1 V(i-1,k_1+1,k_2) + \alpha_2 V(i-1,k_1,k_2+1) + (ir_0+k_1r_1+k_2r_2)u \\ &\geq \alpha_1\left[V(i-1,k_1,k_2)+c_1\right] + \alpha_2\left[V(i-1,k_1,k_2)+c_2\right] + (ir_0+k_1r_1+k_2r_2)u \\ &= V(i-1,k_1,k_2) + (\alpha_1c_1+\alpha_2c_2+r_0u) + [(i-1)r_0+k_1r_1+k_2r_2]u \\ &\geq V(i-1,k_1,k_2) + c_0 + [(i-1)r_0+k_1r_1+k_2r_2]\tau_0 = J_{SU}(i,k_1,k_2), \end{aligned}$$

where the last inequality follows from Assumption 1 and the assumption of part (iii) of the lemma. Hence, taking action SU is always at least as good as taking action Tr if $u \geq \tau_0$. □

Proofs of (i)&(ii) of Theorem 1:

Part (i): Let $k = i + k_1 + k_2$. If $k = 1$, the result trivially holds since $k = k_1 = 1$, i.e., the only patient in the system is of class-1. Now assume the result is true for some $k \geq 1$. Using interchange arguments we will show that it holds for $k+1$ as well. One by one, we will show that *serve class 1* (SC1) is better than every other possible action.

(i)-1: Action **SC1** is better than **SC2**

Define policy π_1 as the policy that first serves a class-2 patient (assuming there is one) and then follows the optimal policy. Then, under policy π_1, the second patient served must be of class-1 by the induction assumption. Now consider policy γ_1 that switches the order of the first two patients under policy π_1 and then follows the same set of actions. The expected cost for the two policies are $J_{\pi_1} = c_2 + c_1 + r_1\tau_2 + J_1$ and $J_{\gamma_1} = c_1 + c_2 + r_2\tau_1 + J_1$, where J_1 denotes the expected waiting cost and service cost for the remaining $(k-1)$ patients, which is the same under both policy π_1 and γ_1. Then, $J_{\gamma_1} - J_{\pi_1} = r_2\tau_1 - r_1\tau_2 = (r_2/\tau_2 - r_1/\tau_1)\,\tau_1\tau_2 \leq 0$. Hence, serving a class-1 patient is no worse than serving a class-2 patient.

(i)-2: Action **SC1** is better than **SU**

Define policy π_2 as the policy that first serves a class-0 patient (assuming there is one) without triage and then follows the optimal policy. Then, under policy π_2, the second patient served must be of class-1. Now consider policy γ_2 that switches the order of the first two patients under policy π_2 and then follows the same set of actions. The expected cost under the two policies are respectively $J_{\pi_2} = c_0 + c_1 + r_1\tau_0 + J_2$ and $J_{\gamma_2} = c_1 + c_0 + r_0\tau_1 + J_2$, where J_2 denotes the expected waiting cost and service cost for the remaining $k-1$ patients, which is the same under both policy π_2 and γ_2. Then, $J_{\gamma_2} - J_{\pi_2} = r_0\tau_1 - r_1\tau_0 \leq 0$. Hence, serving the class 1 first is no worse than serving without triage.

(i)-3: Action **SC1** is better than **Tr**

Define policy $\pi_3(m)$ as the policy that first triages m class-0 patients (where $0 \leq m \leq i$) then serves one class 1 patient and follows the optimal policy. From (i)-1 and (i)-2 above, we know that π_{m^*} is optimal for some $0 \leq m^* \leq i$. If $m^* = 0$, the proof is done. Otherwise, consider policy $\gamma_3(m)$, $1 \leq m \leq i$, which triages $m-1$ patients first, then serves a class-1 patient, performs one more triage and then follows the optimal policy. The expected cost of policy $\pi_3(m)$ and $\gamma_3(m)$ are $J_{\pi_3(m)} = \psi + (r_0+r_1)u + c_1 + \alpha_1\cdot r_1\tau_1 + \alpha_2\cdot r_2\tau_1 + J_3$ and $J_{\gamma_3(m)} = \psi + c_1 + r_0(\tau_1+u) + J_3$, where ψ denotes the expected cost incurred during the triage of the first $(m-1)$ patients, J_3 denotes the expected waiting cost and service cost that will incur after the service of the first class-1 patient in $\pi_3(m)$ and after the completion of serving the mth class-0 patient in $\gamma_3(m)$. Note that these costs are the same in both policy $\pi_3(m)$ and $\gamma_3(m)$. Then, $J_{\gamma_3(m)} - J_{\pi_3(m)} = (r_0 - \alpha_1r_1 - \alpha_2r_2)\tau_1 - r_1u = -r_1u < 0$, $1 \leq m \leq i$. Hence, policy $\pi_3(m)$ is outperformed by policy $\gamma_3(m)$ and can not be the optimal policy, $1 \leq m \leq i$. Thus, the optimal policy must be $\pi_3(0)$, i.e., the server should first serve a class-1 patient instead of doing triage. This completes the proof of part *(i)*.

Part (ii): If $k_1 > 0$, then $a^*(i,k_1,k_2) = \text{SC1}$ by part (i), hence we only need to consider the case where $k_1 = 0$ and $i > 0$ and $k_2 > 0$. We show that SC2 is not the optimal decision, meaning that either Tr or SU is more preferable than SC2, by induction on the number of remaining patients k, as in the proof of part (i).

Suppose that $k = 2$, i.e. there is one class 2 and one class-0 patient. Consider two policies: policy π serves the class-2 patient, then serves the class-0 patient without triage (since there is only one patient left, doing triage is inferior); policy γ first serves the class-0 patient without triage then serves the class-2 patient. The expected costs for policies π and γ are respectively $J_\pi = c_2 + r_0\tau_2 + c_0$, $J_\gamma = c_0 + r_2\tau_0 + c_2$, and $J_\pi - J_\gamma = r_0\tau_2 - r_2\tau_0 \geq 0$.

Now, assume $a^*(i, 0, k_2) \neq$ SC2 for some $i + k_2 = k \geq 2$. We will show that $a^*(i, 0, k_2) \neq$ SC2 when $i + k_2 = k + 1$. Suppose π first serves a class-2 patient in state $(i, 0, k_2)$, then follows the optimal policy. By the induction hypothesis, in $(i, 0, k_2 - 1)$ the optimal policy will work on a class-0 patient, by either serving without triage (SU) or performing triage (Tr). Consider another policy γ, which, when in state $(i, 0, k_2)$, does whatever π does in $(i, 0, k_2 - 1)$, then serves the class-2 patient that π serves in $(i, 0, k_2)$ and goes on to follow policy π.

If π takes action SU in state $(i, 0, k_2 - 1)$, then the expected cost under policy π and γ are $J_\pi = c_2 + r_0\tau_2 + c_0 + \Gamma_1$ and $J_\gamma = c_0 + r_2\tau_0 + c_2 + \Gamma_1$, where Γ_1 denotes the expected waiting and service cost incurred by the remaining $i - 1$ class-0 and $k_2 - 1$ class-2 patients, which is the same under the two policies. Then, $J_\pi - J_\gamma = r_0\tau_2 - r_2\tau_0 \geq 0$.

If π takes action Tr in state $(i, 0, k_2 - 1)$, i.e., $a^*(i, 0, k_2 - 1) =$ Tr, first, by Lemma 1 we must have $u < \tau_0$. The expected cost under policy π and γ are $J_\pi = c_2 + r_0\tau_2 + r_0 u + \Gamma_2$ and $J_\gamma = r_0 u + r_2 u + c_2 + \Gamma_2$, where Γ_2 denotes the expected waiting and service cost incurred by the remaining patients ($i - 1$ class-0 patients, one patient which has just been triaged, and $k_2 - 1$ already waiting class-2 patients), which is the same under the two policies. Then, $J_\pi - J_\gamma = r_0\tau_2 - r_2 u = r_0\tau_2 - r_2\tau_0 + r_2\tau_0 - r_2 u = (r_0/\tau_0 - r_2/\tau_2)\tau_0\tau_2 + r_2(\tau_0 - u) > 0$. Thus, we can conclude that SC2 is not an optimal action in state $(i, 0, k_2)$ when $i > 0$, i.e., $a^*(i, 0, k_2) \neq$ SC2 as long as $i > 0$. □

To prove Part (iii) of Theorem 1, we first establish a series of lemmas, which eventually lead to the proof of the theorem.

Lemma 2. *Assume $r_0/\tau_0 \geq r_2/\tau_2$. We have $a^*(1, 0, k_2) =$ SU for all $k_2 \geq 0$.*

Proof of Lemma 2: In state $(1, 0, k_2)$, by Theorem 1, SC2 is suboptimal. Hence, the only actions that can be optimal are Tr and SU. Let J_{Tr} denote the expected cost of taking action Tr next then following the optimal policy until all patients are served, and J_{SU} denote the expected cost of taking action SU next then following the optimal policy until all patients are served. Then,

$$\begin{aligned} J_{Tr}(1, 0, k_2) &= \alpha_1 V(0, 1, k_2) + \alpha_2 V(0, 0, k_2 + 1) + (r_0 + k_2 r_2)u \\ &= \alpha_1(V(0, 0, k_2) + c_1 + k_2 r_2 \tau_1) + \alpha_2(V(0, 0, k_2) + c_2 + k_2 r_2 \tau_2) + (r_0 + k_2 r_2)u \\ &= V(0, 0, k_2) + (\alpha_1 c_1 + \alpha_2 c_2 + r_0 u) + k_2 r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u), \\ J_{SU}(1, 0, k_2) &= V(0, 0, k_2) + c_0 + k_2 r_2 \tau_0. \end{aligned}$$

By Assumption 1, $J_{Tr}(1, 0, k_2) - J_{SU}(1, 0, k_2) = (\alpha_1 c_1 + \alpha_2 c_2 + r_0 u - c_0) + k_2 r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0$. □

Lemma 3. *Assume $r_0/\tau_0 \geq r_2/\tau_2$. For all $(i, k_1, k_2) \in \mathcal{S}$, we have*

$$V(i, k_1, k_2 + 1) - V(i, k_1, k_2) \geq c_2 + r_2(i\tau_0 + k_1\tau_1 + k_2\tau_2).$$

Proof of Lemma 3: By Theorem 1 parts (i)&(ii), we know that under the optimal policy, first, all class-1 patients are served. Therefore,

$$V(i, k_1, k_2 + 1) - V(i, k_1, k_2) = V(i, 0, k_2 + 1) - V(i, 0, k_2) + r_2 k_1 \tau_1. \tag{9}$$

Define $\tilde{V}(i, 0, k_2)$ for any $i + k_2 \geq 1$ as the total expected cost, starting from state $(i, 0, k_2)$, under the policy that uses the action that is optimal for state $(\tilde{i}, 0, \tilde{k}_2 + 1)$ whenever the system state is $(\tilde{i}, 0, \tilde{k}_2)$. Thus, $\tilde{V}(i, 0, k_2) \geq V(i, 0, k_2)$ and therefore

$$\begin{aligned} V(i, 0, k_2 + 1) - V(i, 0, k_2) &= V(i, 0, k_2 + 1) - \tilde{V}(i, 0, k_2) + \tilde{V}(i, 0, k_2) - V(i, 0, k_2) \\ &\geq V(i, 0, k_2 + 1) - \tilde{V}(i, 0, k_2). \end{aligned} \tag{10}$$

The only difference between $V(i,0,k_2+1)$ and $\tilde{V}(i,0,k_2)$ is the expected cost incurred by the extra class-2 patient in the former, which includes the expected service cost plus the expected waiting cost during the service of the previous $i+k_2$ patients. By Assumption 1, the expected time for serving a patient without triage, which is τ_0, is less than that for first triaging then serving this patient, which is $u+\alpha_1\tau_1+\alpha_2\tau_2$. Hence, the expected waiting time of the last class-2 patient is greater than or equal to $i\tau_0+k_2\tau_2$. Therefore,

$$V(i,0,k_2+1)-\tilde{V}(i,0,k_2)\geq c_2+r_2(i\tau_0+k_2\tau_2). \tag{11}$$

Combining (9), (10) and (11), $V(i,k_1,k_2+1)-V(i,k_1,k_2)\geq c_2+r_2(i\tau_0+k_1\tau_1+k_2\tau_2)$. □

Lemma 4. *Assume $r_0/\tau_0\geq r_2/\tau_2$. If $u\geq\tilde{u}=\alpha_1(r_1\tau_0-r_0\tau_1)/r_0$, then $a^*(i,k_1,k_2)\neq \mathrm{Tr}$ for any $(i,k_1,k_2)\in\mathcal{S}$.*

Proof of Lemma 4: Suppose the current state is (i,k_1,k_2). It is sufficient to consider the case $k_1=0$ and $i>0$, because $a^*(i,k_1,k_2)=\mathrm{SC1}$ when $k_1>0$ and Tr is not a feasible action when $i=0$.

Suppose $k_1=0,\ i>0$. Theorem 1 says that SC2 is suboptimal. Hence, the only possible optimal actions are Tr and SU. Let J_{Tr} denote the expected cost of choosing Tr first and then using the optimal policy until all patients are served, and J_{SU} denote the expected cost of choosing SU first then using the optimal policy until all patients are served. Thus, we have

$$\begin{aligned}
J_{Tr}(i,0,k_2)&=\alpha_1V(i-1,1,k_2)+\alpha_2V(i-1,0,k_2+1)+(ir_0+k_2r_2)u\\
&=\alpha_1[V(i-1,0,k_2)+c_1+(i-1)r_0\tau_1+k_2r_2\tau_1]+\alpha_2V(i-1,0,k_2+1)+(ir_0+k_2r_2)u,\\
J_{SU}(i,0,k_2)&=V(i-1,0,k_2)+c_0+[(i-1)r_0+k_2r_2]\tau_0.
\end{aligned}$$

Then, from Lemma 3,

$$\begin{aligned}
&J_{Tr}(i,0,k_2)-J_{SU}(i,0,k_2)\\
=&\alpha_2\left[V(i-1,0,k_2+1)-V(i-1,0,k_2)\right]+\alpha_1[c_1+(i-1)r_0\tau_1+k_2r_2\tau_1]+(ir_0+k_2r_2)u\\
&-c_0-[(i-1)r_0+k_2r_2]\tau_0\\
\geq&\alpha_2[c_2+(i-1)r_2\tau_0+k_2r_2\tau_2]+\alpha_1[c_1+(i-1)r_0\tau_1+k_2r_2\tau_1]+(ir_0+k_2r_2)u-c_0-[(i-1)r_0+k_2r_2]\tau_0\\
=&(\alpha_1c_1+\alpha_2c_2+r_0u-c_0)+k_2r_2(\alpha_1\tau_1+\alpha_2\tau_2+u-\tau_0)+(i-1)(\alpha_1r_0\tau_1+\alpha_2r_2\tau_0+r_0u-r_0\tau_0)\\
=&(\alpha_1c_1+\alpha_2c_2+r_0u-c_0)+k_2r_2(\alpha_1\tau_1+\alpha_2\tau_2+u-\tau_0)+(i-1)r_0(u-\tilde{u})>0.
\end{aligned}$$

The last inequality is by Assumption 1 and the lemma assumption that $u\geq\tilde{u}$. Hence, $a^*(i,k_1,k_2)\neq$ Tr, $\forall\,(i,k_1,k_2)\in\mathcal{S}$. □

Lemma 5. *Assume $r_0/\tau_0\geq r_2/\tau_2$. If $a^*(i,0,k_2)=\mathrm{SU}$, then $a^*(j,0,k_2)=\mathrm{SU}$ for all $1\leq j\leq i$.*

Proof of Lemma 5: If $u\geq\tilde{u}$, by parts (i) and (ii) of Theorem 1 and Lemma 4, we know $a^*(i,0,k_2)=\mathrm{SU}$ for all $i\geq1,k_2\geq0$. Thus, the lemma holds trivially.

Let us now assume that $u<\tilde{u}$. Let policy π be the policy that first serves the i class-0 patients without triage, then serves the k_2 class-2 patients, and J_π denotes the expected total cost under policy π. Assume that policy π is not optimal, then there must exist at least one policy that does better than π and satisfies the following properties: *The policy first serves $1\leq k\leq i-1$ class-0 patients without triage, then performs triage for the next class-0 patient. And in conformance with the properties of the optimal policy as established in Theorem 1, if the patient that goes through triage is classified as class-1, γ_1 serves that patient right away. Otherwise, the patient is served at the end together with all the other class-2 patients.* Suppose that among the policies which satisfy these properties, γ_1 is the policy for which k is the smallest, and let $k_{\min}$ denote that smallest value for k, i.e., γ_1 first serves $k_{\min}$ class-0 patients without triage, then performs triage for the next patient. Note that by definition, we have $J_{\gamma_1}<J_\pi$, where J_{γ_1} is the total expected cost under policy γ_1.

Now, consider another policy γ_2, which serves $k_{\min}-1$ class-0 patients without triage, performs triage on the next patient, serves the next patient without triage, and then takes the same actions as γ_1. As in γ_1, if triage results in identification of a class-1 patient, that patient is served immediately; otherwise, the patient is served at the end with all the other class-2 patients. Thus, the only difference between γ_1 and γ_2 is that while γ_1 serves the $k_{\min}$th class-0 patient without triage and triages the $(k_{\min}+1)$th class-0 patient, γ_2 triages the $k_{\min}$th class-0 patient and serves the $(k_{\min}+1)$th class-0 patient without triage. Since by definition, policy γ_1 is the one with the smallest k among those policies that perform better than policy π, we have

$$J_{\gamma_2} \geq J_\pi > J_{\gamma_1}. \tag{12}$$

If the only triaged patient among the first $k_{\min}+1$ patients is of class-1, let Γ_1 denote the total expected cost that will incur after the triage and service of the $(k_{\min}+1)$th patient in policy γ_1 (or service without triage of the $(k_{\min}+1)$th patient in policy γ_2). If the only triaged patient is of class-2, denote Γ_2 as the total expected cost that will incur after the triage of the $(k_{\min}+1)$th patient in policy γ_1 (or service without triage of the $(k_{\min}+1)$th patient in policy γ_2). The total expected cost under policy γ_1 and γ_2 are respectively

$$\begin{aligned}
J_{\gamma_1} &= J + c_0 + [(i-k_{\min})r_0 + k_2r_2]\,\tau_0 + [(i-k_{\min})r_0 + k_2r_2]\,u \\
&\quad + \alpha_1\,[c_1 + (i-k_{\min}-1)r_0\tau_1 + k_2r_2\tau_1 + \Gamma_1] + \alpha_2\Gamma_2, \\
J_{\gamma_2} &= J + [(i-k_{\min}+1)r_0 + k_2r_2]\,u \\
&\quad + \alpha_1\,[c_1 + (i-k_{\min})r_0\tau_1 + k_2r_2\tau_1 + c_0 + (i-k_{\min}-1)r_0\tau_0 + k_2r_2\tau_0 + \Gamma_1] \\
&\quad + \alpha_2\,[c_0 + (i-k_{\min}-1)r_0\tau_0 + (k_2+1)r_2\tau_0 + \Gamma_2],
\end{aligned}$$

where J is the total expected cost to be incurred during the service of the first $(k_{\min}-1)$ class-0 patients without triage. Then,

$$J_{\gamma_1} - J_{\gamma_2} = -r_0u + r_0\tau_0 - \alpha_1r_0\tau_1 - \alpha_2r_2\tau_0 = -r_0u + r_0(\tau_0 - \frac{\alpha_1r_0\tau_1 + \alpha_2r_2\tau_0}{r_0}) = r_0(\tilde{u} - u) > 0.$$

Hence, $J_{\gamma_2} < J_{\gamma_1}$, which is a contradiction to (12). □

Lemma 6. *Assume $r_0/\tau_0 \geq r_2/\tau_2$.*

(i) If $a^*(i,0,k_2) = \mathrm{SU}$, then $a^*(\tilde{i},0,\tilde{k}_2) = \mathrm{SU}$ for any $1 \leq \tilde{i} \leq i$ and $k_2 \leq \tilde{k}_2 \leq N-\tilde{i}$.

(ii) If $a^*(i,0,k_2) = \mathrm{Tr}$, then $a^*(\tilde{i},0,\tilde{k}_2) = \mathrm{Tr}$ for any $0 \leq \tilde{k}_2 \leq k_2$ and $i \leq \tilde{i} \leq N-\tilde{k}_2$.

Proof of Lemma 6: Part (i): If $u \geq \tilde{u}$, by Theorem 1 parts (i)&(ii) and Lemma 4 we know $a^*(i,0,k_2) = \mathrm{SU}$ for all $i \geq 1, k_2 \geq 0$. Then, the result is immediate. Now, assume that $u < \tilde{u}$. We will use an induction argument to show that if $a^*(i,0,k_2) = \mathrm{SU}$, then $a^*(j,0,k) = \mathrm{SU}$ for any $1 \leq j \leq i$ and $k_2 \leq k \leq N-j$. When $i=1$, the result holds since $a^*(1,0,k_2) = \mathrm{SU}$ for any $k_2 \geq 0$ by Lemma 2.

Now, for induction, we assume that if $a^*(i-1,0,k_2) = \mathrm{SU}$, then $a^*(j,0,k) = \mathrm{SU}$ for any $1 \leq j \leq i-1$ and $k_2 \leq k \leq N-j$, where $i \geq 2$. Next we consider $(i,0,k_2)$. From the lemma assumption, we have $a^*(i,0,k_2) = \mathrm{SU}$. Then, by Lemma 5, we know that $a^*(j,0,k_2) = \mathrm{SU}$ for any $1 \leq j \leq i$, especially, $a^*(i-1,0,k_2) = \mathrm{SU}$. By the induction assumption, $a^*(j,0,k) = \mathrm{SU}$ for any $1 \leq j \leq i-1$ and $k_2 \leq k \leq N-j$. It remains to show that $a^*(i,0,k) = \mathrm{SU}$ for any $k_2 \leq k \leq N-i$. Let $J_{Tr}(i,k)$ denote the total expected cost of performing triage in state $(i,0,k)$ and then following the optimal policy. Then, $J_{Tr}(i,k) = (ir_0 + kr_2)u + \alpha_1V(i-1,1,k) + \alpha_2V(i-1,0,k+1)$.

Similarly, let $J_{SU}(i,k)$ denote the total expected cost for choosing to serve a class-0 patient in state $(i,0,k)$ and then following the optimal policy. Therefore, $J_{SU}(i,k) = c_0 + [(i-1)r_0 + kr_2]\tau_0 + V(i-1,0,k)$. Using Theorem 1 parts (i)&(ii) and the induction assumption,

$$\begin{aligned}
V(i-1,1,k) &= c_1 + [(i-1)r_0 + kr_2]\tau_1 + V(i-1,0,k), \\
V(i-1,0,k+1) &= c_2 + r_2[(i-1)\tau_0 + k\tau_2] + V(i-1,0,k).
\end{aligned}$$

Plugging them into the expression for $J_{Tr}(i,k)$,

$$\begin{aligned}
J_{Tr}(i,k) = (ir_0 + kr_2)u &+ \alpha_1\,(c_1 + [(i-1)r_0 + kr_2]\tau_1 + V(i-1,0,k)) \\
&+ \alpha_2\,(c_2 + r_2[(i-1)\tau_0 + k\tau_2] + V(i-1,0,k))\,.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&J_{SU}(i, k_2+1) - J_{Tr}(i, k_2+1) \\
&= c_0 + [(i-1)r_0 + kr_2]\tau_0 - (ir_0 + kr_2)u - \alpha_1\left(c_1 + [(i-1)r_0 + kr_2]\tau_1\right) - \alpha_2\left(c_2 + r_2[(i-1)\tau_0 + k\tau_2]\right) \\
&= (c_0 - \alpha_1 c_1 - \alpha_2 c_2 - r_0 u) + (i-1)r_0(\tilde{u} - u) - kr_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0).
\end{aligned}
$$

We know that $a^*(i, 0, k_2) = \text{SU}$, which implies $J_{SU}(i, k_2) - J_{Tr}(i, k_2) \leq 0$. Then,

$$
[J_{SU}(i, k) - J_{Tr}(i, k)] - [J_{SU}(i, k_2) - J_{Tr}(i, k_2)] = -(k - k_2)(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) \leq 0,
$$

which implies that $a^*(i, 0, k) = \text{SU}$ for any $k_2 \leq k \leq N - i$.

Part (ii): Given part *(i)*, the proof of *(ii)* is trivial. Assume $a^*(i, 0, k_2) = \text{Tr}$, and there exists $\bar{i} > i$ (or $\bar{k}_2 < k_2$) such that $a^*(\bar{i}, 0, k_2) = \text{SU}$ (or $a^*(i, 0, \bar{k}_2) = \text{SU}$), which is a direct contradiction to *(i)*, which implies that $a^*(i, 0, k_2) = \text{SU}$. □

Proof of Theorem 1 Part (iii): First, note that if $u > \tilde{u}$, then by Lemma 4 and Theorem 1 parts (i)&(ii), the optimal action in all states is SU. One can check to see that when $u > \tilde{u}$, $L(\cdot)$ has a negative slope and thus the theorem trivially holds. Hence, in the following, it is sufficient to consider the case where $u \leq \tilde{u}$.

By Theorem 1, we can write the system states $\{(i, 0, k_2) : i \geq 1, k_2 \geq 0\}$ as the union of the following three disjoint sets:

$$
\begin{aligned}
&\mathcal{S}_1 = \{(i, 0, k_2) : a^*(i, 0, k_2) = \text{SU}\}, \\
&\mathcal{S}_2 = \{(i, 0, k_2) : a^*(i, 0, k_2) = \text{Tr},\ a^*(i-1, 0, k_2) = \text{SU}\}, \\
&S_3 = \{(i, 0, k_2) : a^*(i, 0, k_2) = \text{Tr},\ a^*(i-1, 0, k_2) = \text{Tr}\}.
\end{aligned}
$$

We show that all the states in $\mathcal{S}_1$ reside above $L(i)$ and all the states in $\mathcal{S}_2, \mathcal{S}_3$ reside below $L(i)$.

First, suppose that $(i, 0, k_2) \in \mathcal{S}_1$. Consider a policy γ that serves a class-0 patient without triage in $(i, 0, k_2)$, then follows the optimal policy. Consider another policy π that performs triage in $(i, 0, k_2)$ then follows the optimal policy. The total expected costs (respectively, $J_\gamma(i, 0, k_2)$ and $J_\pi(i, 0, k_2)$) can be written as follows:

$$
\begin{aligned}
J_\gamma(i, 0, k_2) &= c_0 + [(i-1)r_0 + k_2 r_2]\tau_0 + V(i-1, 0, k_2), \\
J_\pi(i, 0, k_2) &= \alpha_1 V(i-1, 1, k_2) + \alpha_2 V(i-1, 0, k_2+1) + (ir_0 + k_2 r_2)u \\
&= \alpha_1\Big(c_1 + [(i-1)r_0 + k_2 r_2]\,\tau_1 + V(i-1, 0, k_2)\Big) + \alpha_2 V(i-1, 0, k_2+1) + (ir_0 + k_2 r_2)u \\
&= \alpha_1\Big(c_1 + [(i-1)r_0 + k_2 r_2]\,\tau_1 + V(i-1, 0, k_2)\Big) + (ir_0 + k_2 r_2)u \\
&\quad + \alpha_2\Big(V(i-1, 0, k_2) + c_2 + r_2\,[(i-1)\tau_0 + k_2\tau_2]\Big) \\
&= V(i-1, 0, k_2) + (ir_0 + k_2 r_2)u + \alpha_1 c_1 + \alpha_2 c_2 + \alpha_1\,[(i-1)r_0 + k_2 r_2]\,\tau_1 + \alpha_2 r_2\,[(i-1)\tau_0 + k_2\tau_2]\,.
\end{aligned}
$$

Since γ is optimal,

$$
\begin{aligned}
&J_\pi(i, 0, k_2) - J_\gamma(i, 0, k_2) \\
&= k_2 r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) + (\alpha_1 c_1 + \alpha_2 c_2 + r_0 u - c_0) + (i-1)(\alpha_1 r_0\tau_1 + \alpha_2 r_2\tau_0 + r_0 u - r_0\tau_0) \\
&= k_2 r_2(u - \tilde{\tau}) + (r_0 u - \tilde{c}) - (i-1)r_0(\tilde{u} - u) \geq 0,
\end{aligned}
$$

i.e.,

$$
k_2 \geq \frac{r_0(\tilde{u} - u)}{r_2(u - \tilde{\tau})}i - \frac{r_0\tilde{u} - \tilde{c}}{r_2(u - \tilde{\tau})}.
$$

Now, suppose that $(i, 0, k_2) \in \mathcal{S}_2$. Then, with γ and π exactly as defined above, policy π is the optimal policy, and thus $J_\pi(i, 0, k_2) - J_\gamma(i, 0, k_2) < 0$, i.e.,

$$
k_2 < \frac{r_0(\tilde{u} - u)}{r_2(u - \tilde{\tau})}i - \frac{r_0\tilde{u} - \tilde{c}}{r_2(u - \tilde{\tau})}.
$$

Finally, consider a state $(i, 0, k_2) \in \mathcal{S}_3$. Then, from Lemma 6, we know that there exists $\bar{i} < i$ such that $(\bar{i}, 0, k_2) \in \mathcal{S}_2$. Then since $\tilde{u} - u \geq 0$ and we know that, as established above,

$$k_2 < \frac{r_0(\tilde{u} - u)}{r_2(u - \tilde{\tau})} i - \frac{r_0\tilde{u} - \tilde{c}}{r_2(u - \tilde{\tau})},$$

we must have

$$k_2 < \frac{r_0(\tilde{u} - u)}{r_2(u - \tilde{\tau})} i - \frac{r_0\tilde{u} - \tilde{c}}{r_2(u - \tilde{\tau})},$$

which completes the proof. □

5 Proof of Proposition 3.

Part (i): Policy *NT* is optimal if and only if either $u \geq \tilde{u}$ (by Lemma 4) or $L(\cdot)$ has a positive slope but the x-intercept of $L(\cdot)$, denoted by x_{int}, is greater than N (by Theorem 1), which can be written as $x_{int} = \frac{r_0\tilde{u}-\tilde{c}}{r_0(\tilde{u}-u)} \geq N$, or equivalently $u \geq u_1 = \tilde{u} - \frac{r_0\tilde{u}-\tilde{c}}{Nr_0}$. Then, the result follows.
Part (ii): From Theorems 1, we know that *TP1* policy is optimal if and only if $u \leq \tilde{u}$, $1 \leq x_{int} \leq 2$, and the x-coordinate of the intersection of line $L(\cdot)$ and the line expressed by $i + k_2 = N$ (where i is the number of class-0 patients and k_2 is the number of class-2 patients) is also between 1 and 2. The last two conditions can be expressed as $1 \leq \frac{r_0\tilde{u}-\tilde{c}}{r_0(\tilde{u}-u)} \leq 2$ and $1 \leq \frac{Nr_2(u-\tilde{\tau})+r_0\tilde{u}-\tilde{c}}{r_2(u-\tilde{\tau})+r_0(\tilde{u}-u)} \leq 2$, respectively. First, if $u \leq \tilde{u}$, then $1 \leq \frac{r_0\tilde{u}-\tilde{c}}{r_0(\tilde{u}-u)}$. It also follows that $1 \leq \frac{Nr_2(u-\tilde{\tau})+r_0\tilde{u}-\tilde{c}}{r_2(u-\tilde{\tau})+r_0(\tilde{u}-u)}$. When $u \leq \tilde{u}$, the condition $\frac{r_0\tilde{u}-\tilde{c}}{r_0(\tilde{u}-u)} \leq 2$ can equivalently be written as $u \leq \frac{r_0\tilde{u}+\tilde{c}}{2r_0}$ and the condition $\frac{Nr_2(u-\tilde{\tau})+r_0\tilde{u}-\tilde{c}}{r_2(u-\tilde{\tau})+r_0(\tilde{u}-u)} \leq 2$ can equivalently be written as $u \leq u_2 = \frac{r_0\tilde{u}+\tilde{c}+(N-2)r_2\tilde{\tau}}{2r_0+(N-2)r_2}$, which completes the proof.
Part (iii): If $r_0\tilde{u} - \tilde{c} > 0$, then $u_1 = \tilde{u} - \frac{r_0\tilde{u}-\tilde{c}}{Nr_0}$ is increasing in N; otherwise, $u_1 = \tilde{u}$ which does not change with N. If $2r_0\tilde{\tau} - r_0\tilde{u} - \tilde{c} \geq 0$, then

$$\frac{r_0\tilde{u} + \tilde{c} + (N-2)r_2\tilde{\tau}}{2r_0 + (N-2)r_2} \geq \frac{r_0\tilde{u} + \tilde{c}}{2r_0},$$

hence, $u_2 = \frac{r_0\tilde{u}+\tilde{c}}{2r_0}$ which does not change with N. Otherwise,

$$u_2 = \frac{r_0\tilde{u} + \tilde{c} + (N-2)r_2\tilde{\tau}}{2r_0 + (N-2)r_2} = \tilde{\tau} - \frac{2r_0\tilde{\tau} - r_0\tilde{u} - \tilde{c}}{2r_0 + (N-2)r_2},$$

which obviously is decreasing with N. This completes the proof. □

6 Proof of Theorem 2.

Step 1: We first show, by induction on the total number of remaining patients, that, serving a class-1 patient is preferable to other possible actions when $k_1 \geq 1$. Let $k = i + k_1 + k_2$. If $k = 1$, the result trivially holds since $k = k_1 = 1$, i.e., the only patient in the system is of class-1. Now assume the result is true for some $k \geq 1$. Using interchange arguments we will show that it holds for $k + 1$ as well. One by one, we will show that *serve class 1* (SC1) is better than every other possible action.

(i)-1: Action **SC1** is better than **SC2**

Define policy π_1 as the policy that first serves a class-2 patient (assuming there is one) and then follows the optimal policy. Then, under policy π_1, the second patient served must be of class-1 by the induction assumption. Now consider policy γ_1 that switches the order of the first two patients under policy π_1 and then follows the same set of actions. The expected cost of the two policies are $J_{\pi_1} = c_2 + c_1 + r_1\tau_2 + J_1$ and $J_{\gamma_1} = c_1 + c_2 + r_2\tau_1 + J_1$, where J_1 denotes the expected waiting cost and service cost for the remaining $(k-1)$ patients, which is the same under both policy π_1 and γ_1. Then, $J_{\gamma_1} - J_{\pi_1} = r_2\tau_1 - r_1\tau_2 = \tau_1\tau_2(r_2/\tau_2 - r_1/\tau_1) \leq 0$. Hence, serving a class-1 patient is better than serving a class-2 patient.

(i)-2: Action **SC1** is better than **SU**

Define policy π_2 as the policy that first serves an untriaged patient (assuming there is one) without triage and then follows the optimal policy. Then, under policy π_2, the second patient served must be of class-1. Now consider policy γ_2 that switches the order of the first two patients under policy π_2 and then follows the same set of actions. The expected cost under the two policies are respectively $J_{\pi_2} = c_0 + c_1 + r_1\tau_0 + J_2$ and $J_{\gamma_2} = c_1 + c_0 + r_0\tau_1 + J_2$, where J_2 denotes the expected waiting cost and service cost for the remaining $k-1$ patients, which is the same under both policy π_2 and γ_2. Then, $J_{\gamma_2} - J_{\pi_2} = r_0\tau_1 - r_1\tau_0 = \tau_0\tau_1(r_0/\tau_0 - r_1/\tau_1) < 0$. Hence, serving a class-1 patient first is better than serving a class-0 patient without triage.

(i)-3: Action **SC1** is better than **Tr**

Define policy $\pi_3(m)$ as the policy that first triages m class-0 patients (where $0 \le m \le i$) then serves one class-1 patient and follows the optimal policy. From (i)-1 and (i)-2 above, we know that π_{m^*} is optimal for some $0 \le m^* \le i$. If $m^* = 0$, the proof is done. Otherwise, consider policy $\gamma_3(m)$, $1 \le m \le i$, which triages $m-1$ patients first, then serves a class-1 patient, performs one more triage and then follows the optimal policy. The expected cost of policy $\pi_3(m)$ and $\gamma_3(m)$ are $J_{\pi_3(m)} = \psi + (r_0 + r_1)u + c_1 + \alpha_1 \cdot r_1\tau_1 + \alpha_2 \cdot r_2\tau_1 + J_3$ and $J_{\gamma_3(m)} = \psi + c_1 + r_0(\tau_1 + u) + J_3$, where ψ denotes the expected cost incurred during the triage of the first $(m-1)$ patients, J_3 denotes the expected waiting cost and service cost that will incur after the service of the first class-1 patient in $\pi_3(m)$ and after the completion of serving the mth class-0 patient in $\gamma_3(m)$. Note that these costs are the same in both policy $\pi_3(m)$ and $\gamma_3(m)$.

$$J_{\gamma_3(m)} - J_{\pi_3(m)} = (r_0 - \alpha_1 r_1 - \alpha_2 r_2)\tau_1 - r_1 u = -r_1 u < 0, \ 1 \le m \le i.$$

Hence, policy $\pi_3(m)$ is outperformed by policy $\gamma_3(m)$ and can not be the optimal policy, $1 \le m \le i$. Thus, the optimal policy must be $\pi_3(0)$, i.e., the server should first serve a class-1 patient instead of doing triage. This completes the proof of Step 1.

Step 2: It is sufficient to consider the case $k_1 = 0$ because $a^*(i, k_1, k_2) = \text{SC1}$ when $k_1 \ge 1$. It is also obvious that $a^*(0, 0, k_2) = \text{SC2}$, $k_2 \ge 1$. Hence, in this step, we only consider the case where $k_1 = 0$, $i \ge 1$, $k_2 \ge 0$. We show that $a^*(i, 0, 0) = \text{SU}$ when $i \ge 1$ and $a^*(i, 0, k_2) = \text{SC2}$ when $i \ge 1$, $k_2 \ge 1$ by induction on the number of remaining class-0 patients i.

We first show that the statement is true for $i = 1$. When $i = 1$ and $k_2 = 0$, the total expected cost of first doing *triage* then following the optimal policy is $J_{Tr}(1, 0, 0) = \alpha_1 V(0, 1, 0) + \alpha_2 V(0, 0, 1) + r_0 u = \alpha_1 c_1 + \alpha_2 c_2 + r_0 u$. The total expected cost of first doing SU and then following the optimal policy is $J_{SU}(1, 0, 0) = c_0$. By Assumption 1 we have $J_{Tr}(1, 0, 0) > J_{SU}(1, 0, 0)$, hence $a^*(1, 0, 0) = \text{SU}$.

When $i = 1$ and $k_2 = 1$, the total expected cost of first doing *triage* then following the optimal policy is

$$\begin{aligned} J_{Tr}(1, 0, 1) &= \alpha_1 V(0, 1, 1) + \alpha_2 V(0, 0, 2) + (r_0 + r_2)u \\ &= \alpha_1[c_1 + r_2\tau_1 + c_2] + \alpha_2[c_2 + r_2\tau_2 + c_2] + (r_0 + r_2)u \\ &= (r_0 u + \alpha_1 c_1 + \alpha_2 c_2) + r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u) + c_2. \end{aligned}$$

The total expected cost of first doing SU and then following the optimal policy is $J_{SU}(1, 0, 1) = c_0 + r_2\tau_0 + c_2$. The total expected cost of first doing SC2 and then following the optimal policy is $J_{SC2}(1, 0, 1) = c_2 + r_0\tau_2 + c_0$. By Assumption 1,

$$J_{Tr}(1, 0, 1) - J_{SU}(1, 0, 1) = (r_0 u + \alpha_1 c_1 + \alpha_2 c_2 - c_0) + r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0.$$
$$J_{SU}(1, 0, 1) - J_{SC2}(1, 0, 1) = r_2\tau_0 - r_0\tau_2 = \tau_0\tau_2(r_2/\tau_2 - r_0/\tau_0) > 0.$$

We have $J_{Tr}(1, 0, 1) > J_{SU}(1, 0, 1) > J_{SC2}(1, 0, 1)$, hence $a^*(1, 0, 1) = \text{SC2}$. Next, we use induction on k_2 to show that $a^*(1, 0, k_2) = \text{SC2}$ for all $k_2 \ge 1$. Assume it is true for $k_2 - 1$ where $k_2 \ge 2$, consider $(i, 0, k_2)$. The total expected cost of first doing Tr then following the optimal policy is

$$\begin{aligned} J_{Tr}(1,0,k_2) &= \alpha_1 V(0,1,k_2) + \alpha_2 V(0,0,k_2+1) + (r_0 + k_2 r_2)u \\ &= \alpha_1[c_1 + k_2 r_2 \tau_1 + V(0,0,k_2)] + \alpha_2[c_2 + k_2 r_2 \tau_2 + V(0,0,k_2)] + (r_0 + k_2 r_2)u \\ &= (r_0 u + \alpha_1 c_1 + \alpha_2 c_2) + k_2 r_2(\alpha_1 \tau_1 + \alpha_2 \tau_2 + u) + V(0,0,k_2). \end{aligned}$$

The total expected cost of first doing SU and then following the optimal policy is

$$J_{SU}(1,0,k_2) = c_0 + k_2 r_2 \tau_0 + V(0,0,k_2) = c_0 + k_2 r_2 \tau_0 + k_2 c_2 + k_2(k_2-1) r_2 \tau_2/2.$$

The total expected cost of first doing SC2 and then following the optimal policy is

$$\begin{aligned} J_{SC2}(1,0,k_2) &= c_2 + [r_0 + (k_2-1)r_2]\tau_2 + V(1,0,k_2-1) \\ &= c_2 + [r_0 + (k_2-1)r_2]\tau_2 + (k_2-1)c_2 + (k_2-1)(k_2-2)r_2\tau_2/2 + r_0(k_2-1)\tau_2 + c_0. \end{aligned}$$

By Assumption 1,

$$J_{Tr}(1,0,k_2) - J_{SU}(1,0,k_2) = (r_0 u + \alpha_1 c_1 + \alpha_2 c_2 - c_0) + k_2 r_2(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0.$$
$$J_{SU}(1,0,k_2) - J_{SC2}(1,0,k_2) = k_2 r_2 \tau_0 - k_2 r_0 \tau_2 = k_2 \tau_0 \tau_2 (r_2/\tau_2 - r_0/\tau_0) > 0.$$

Thus, we have $J_{Tr}(1,0,k_2) > J_{SU}(1,0,k_2) > J_{SC2}(1,0,k_2)$ and $a^*(1,0,k_2) = \text{SC2}$ for any $k_2 \geq 1$.

We now show that $a^*(i,0,0) = \text{SU}$ for any $i \geq 1$ and $a^*(i,0,k_2) = \text{SC2}$ for any $i \geq 1,\ k_2 \geq 1$. Assume the result holds for i where $i \geq 1$, i.e., $a^*(i,0,0) = \text{SU}$ and $a^*(i,0,k_2) = \text{SC2}$ for all $k_2 \geq 1$, we will show it also holds for $i+1$. First consider $(i+1,0,0)$. The total expected cost of first doing Tr then following the optimal policy is

$$\begin{aligned} J_{Tr}(i+1,0,0) &= \alpha_1 V(i,1,0) + \alpha_2 V(i,0,1) + (i+1) r_0 u \\ &= \alpha_1[c_1 + i r_0 \tau_1 + V(i,0,0)] + \alpha_2[c_2 + i r_0 \tau_2 + V(i,0,0)] + (i+1) r_0 u \\ &= (r_0 u + \alpha_1 c_1 + \alpha_2 c_2) + i r_0(\alpha_1\tau_1 + \alpha_2\tau_2 + u) + V(i,0,0). \end{aligned}$$

The total expected cost of first doing SU and then following the optimal policy is $J_{SU}(i+1,0,0) = V(i,0,0) + c_0 + i r_0 \tau_0$. By Assumption 1, $J_{Tr}(i+1,0,0) - J_{SU}(i+1,0,0) = (r_0 u + \alpha_1 c_1 + \alpha_2 c_2 - c_0) + i r_0(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0$. Hence $a^*(i+1,0,0) = \text{SU}$.

Next, we consider $(i+1,0,1)$. The total expected cost of first doing Tr then following the optimal policy is

$$\begin{aligned} J_{Tr}(i+1,0,1) &= \alpha_1 V(i,1,1) + \alpha_2 V(i,0,2) + [(i+1)r_0 + r_2]u \\ &= \alpha_1[c_1 + (i r_0 + r_2)\tau_1 + V(i,0,1)] + \alpha_2[c_2 + (i r_0 + r_2)\tau_2 + V(i,0,1)] + [(i+1)r_0 + r_2]u \\ &= (r_0 u + \alpha_1 c_1 + \alpha_2 c_2) + (i r_0 + r_2)(\alpha_1\tau_1 + \alpha_2\tau_2 + u) + V(i,0,1). \end{aligned}$$

The total expected cost of first doing SU and then following the optimal policy is

$$J_{SU}(i+1,0,1) = c_0 + (i r_0 + r_2)\tau_0 + V(i,0,1) = c_0 + (i r_0 + r_2)\tau_0 + c_2 + i r_0 \tau_2 + i c_0 + i(i-1) r_0 \tau_0/2.$$

The total expected cost of first doing SC2 and then following the optimal policy is

$$J_{SC2}(i+1,0,1) = c_2 + (i+1) r_0 \tau_2 + V(i+1,0,0) = c_2 + (i+1) r_0 \tau_2 + (i+1) c_0 + i(i+1) r_0 \tau_0/2.$$

By Assumption 1,

$$\begin{aligned} & J_{Tr}(i+1,0,1) - J_{SU}(i+1,0,1) \\ =\ & (r_0 u + \alpha_1 c_1 + \alpha_2 c_2 - c_0) + (i r_0 + r_2)(\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0. \\ & J_{SU}(i+1,0,1) - J_{SC2}(i+1,0,1) = r_2\tau_0 - r_0\tau_2 = \tau_0\tau_2(r_2/\tau_2 - r_0/\tau_0) > 0. \end{aligned}$$

We have $J_{Tr}(i+1,0,1) > J_{SU}(i+1,0,1) > J_{SC2}(i+1,0,1)$, hence $a^*(i+1,0,1) = \text{SC2}$. Next, we use induction on k_2 to show that $a^*(i+1,0,k_2) = \text{SC2}$ for all $k_2 \geq 1$. Assume this is true for k_2 where $k_2 \geq 1$, consider $(i+1,0,k_2+1)$. The total expected cost of first doing Tr then following the optimal policy is

$$\begin{aligned}
& J_{Tr}(i+1,0,k_2+1) \\
= \; & \alpha_1 V(i,1,k_2+1) + \alpha_2 V(i,0,k_2+2) + [(i+1)r_0 + (k_2+1)r_2]u \\
= \; & \alpha_1[c_1 + (ir_0 + (k_2+1)r_2)\tau_1 + V(i,0,k_2+1)] + \alpha_2[c_2 + (ir_0 + (k_2+1)r_2)\tau_2 + V(i,0,k_2+1)] \\
& +[(i+1)r_0 + (k_2+1)r_2]u \\
= \; & (r_0 u + \alpha_1 c_1 + \alpha_2 c_2) + [ir_0 + (k_2+1)r_2](\alpha_1\tau_1 + \alpha_2\tau_2 + u) + V(i,0,k_2+1).
\end{aligned}$$

The total expected cost of first doing SU and then following the optimal policy is

$$\begin{aligned}
& J_{SU}(i+1,0,k_2+1) \\
= \; & c_0 + [ir_0 + (k_2+1)r_2]\tau_0 + V(i,0,k_2+1) \\
= \; & c_0 + [ir_0 + (k_2+1)r_2]\tau_0 + (k_2+1)c_2 + k_2(k_2+1)r_2\tau_2/2 + ir_0(k_2+1)\tau_2 + ic_0 + i(i-1)r_0\tau_0/2.
\end{aligned}$$

The total expected cost of first doing SC2 and then following the optimal policy is

$$\begin{aligned}
& J_{SC2}(i+1,0,k_2+1) \\
= \; & c_2 + [(i+1)r_0 + k_2 r_2]\tau_2 + V(i+1,0,k_2) \\
= \; & c_2 + [(i+1)r_0 + k_2 r_2]\tau_2 + k_2 c_2 + k_2(k_2-1)r_2\tau_2/2 + (i+1)r_0 k_2\tau_2 + (i+1)c_0 + i(i+1)r_0\tau_0/2.
\end{aligned}$$

By Assumption 1,

$$\begin{aligned}
& J_{Tr}(i+1,0,k_2+1) - J_{SU}(i+1,0,k_2+1) \\
= \; & (r_0 u + \alpha_1 c_1 + \alpha_2 c_2 - c_0) + [ir_0 + (k_2+1)r_2](\alpha_1\tau_1 + \alpha_2\tau_2 + u - \tau_0) > 0. \\
& J_{SU}(i+1,0,k_2+1) - J_{SC2}(i+1,0,k_2+1) \\
= \; & (k_2+1)(r_2\tau_0 - r_0\tau_2) = (k_2+1)\tau_0\tau_2(r_2/\tau_2 - r_0/\tau_0) > 0.
\end{aligned}$$

We have $J_{Tr}(i+1,0,k_2+1) > J_{SU}(i+1,0,k_2+1) > J_{SC2}(i+1,0,k_2+1)$, hence $a^*(i+1,0,k_2+1) = \text{SC2}$, which completes the proof. □

7 Estimation of Death Probability Functions

To the best of our knowledge, the only work in the emergency medicine literature that estimates the survival probabilities as a function of time are Sacco et al. (2005, 2007) and Navin et al. (2009). These papers provide on-site survival probability estimates for military-age victims with penetrating injuries (a type of trauma that is highly common in armed combat). They categorize patients according to their RPM (Respiratory rate, Pulse rate, Motor response) scores, which takes value from 0 to 12, and lower values are associated with higher criticality, i.e., lower survival probability. Thus, immediate patients would generally have lower RPM scores than delayed patients. While there can be alternative ways of classifying patients as immediate and delayed given their RPM scores, in our numerical study, we follow Mills et al. (2014) and choose RPM scores 4 through 8 as the immediate class and RPM scores 9 through 12 as the delayed class. The survival probabilities for patients with RPM scores 0 through 3 are so low that given the extreme resource limitation we envision here, these patients would be classified as expectant. Then, by averaging the survival probabilities corresponding to different RPM scores and subtracting from 1, we can construct death probability estimates as a function of time for the two triage classes (immediate and delayed). The resulting functions after interpolation and smoothing are given in Figure 2 in the paper.

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