When to Triage in Service Systems with Hidden Customer Class Identities?

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In service systems with heterogeneous customers, prioritization with respect to the relative importance of customers is known to improve certain performance measures. However, in many applications, information necessary to determine the importance level of a customer may not be available immediately but can be revealed only through some preliminary investigation, which is sometimes called *triage*. This triage process is typically error-prone and may take substantial amount of time, and hence, it is not always clear if and when it should be implemented for purposes of priority assignment. To provide insights into this question, we study a stylized queueing model with a single server and two types of customers with hidden type identities, which differ in both their service times and the rates of waiting costs. By means of a Markov decision formulation, we first show that the optimal dynamic policy on triage is characterized by a switching curve. The comparison of two state-independent policies (no-triage and triage-all) shows that the information from triage is more beneficial when the arrival rate is neither too small nor too large. Our numerical results show that as the traffic intensity increases, the sub-optimality of state-independent policies increases especially when the performance of the no-triage policy is similar to that of triage-all policy. Hence, when traffic intensity is light, triage can be bypassed. When traffic intensity is mediocre to high, a more-complex state-dependent triage policy is needed if the two state-independent policies do not differ much in performance; otherwise, the best state-independent policy is fine.

Key words: Priority queues; optimal scheduling; triage; Markov decision processes

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1. Introduction

In many service systems, it is common practice to give priority to a group of customers for service in order to improve key performance measures. Call centers, banks, and hospitals are just a few examples where prioritization of customers is prevalent. For some of these service systems, where the customer base is widely heterogeneous, it is not difficult to see why prioritization should work. For example, emergency departments of hospitals typically receive patients with a large variety of ailments ranging from life-threatening conditions such as heart attack to minor issues such as common cold, and thus, categorizing patients according to their urgency levels and prioritizing accordingly is crucial. Yet, there are many other systems where arriving customers, at first glance, appear to be more or less homogeneous, and thus, it is not clear how prioritization can be implemented or if it would be beneficial at all for them. For these systems, however, there may be ways to put some effort into collecting information from arriving customers to differentiate them from the rest of the population. But then a natural question arises: Is it worthwhile to spend additional time collecting information on a customer before service so as to implement a priority policy, or is it better to stick with a commonly used non-priority policy like first-come-first-served (FCFS)? Of course, the answer to this question would be very much case dependent. In this paper, we aim to shed some light onto this fundamental question by analyzing a stylized mathematical model that captures the trade-off between time lost in identifying priority customers and opportunity lost by not prioritizing them. In particular, we would like to obtain some general insights into conditions under which it is worthwhile spending time to collect information to differentiate customers. We are also interested in the effects of system parameters on the triage decisions such as the arrival rate and the time it takes to collect information.

There are several application areas where insights from such an analysis would be useful – especially within the field of healthcare operations. Consider operations at a radiological service center where radiologists seated in front of computer workstations *process jobs* sequentially. Radiologists are medical doctors specialized in interpreting diagnostic images by X-rays, ultrasound, computed tomography (CT), magnetic resonance imaging (MRI), etc. In most cases, technicians are the ones who meet with patients and take these diagnostic images – not the radiologists. Once the images are taken, they are sent to a radiologist's queues for interpretation. Each job in a radiologist's queue corresponds to a collection of images taken for a patient. As discussed in Ibanez et al. (2018), which uses data from one of the largest outsourced radiological services (teleradiology) firms in the U.S., radiologists may at times use discretion to change the order of jobs in their queue and deviate from FCFS if they believe that reordering jobs would increase their efficiency. However, the authors acknowledge that radiologists spend extra time for a preliminary review of jobs before reordering them and show by an empirical study that the benefits of reordering a queue may not always compensate for the time spent searching and identifying priority jobs. Another similar application is from operations in a genetic testing laboratory, where geneticists interpret patients' test results that arrive continually and queue up at their desks. According to our personal communication with a geneticist (Tolun 2018), similar to radiologists, they also spend additional time on preliminary review of their jobs at times to reorder them based on concerns about urgency and inefficiency.

This information/delay trade-off also arises in the daily operations of emergency departments (EDs) when a physician makes a decision on which patient to treat next among all the patients who have been triaged by nurses and are waiting to be seen by physicians. When a physician becomes available to see a new patient, she will login the ED information system and choose one from the list of patients in the waiting room based on their urgency levels and wait times (see, e.g., Ding et al. 2019). A recent study (Li et al. 2020) finds that when a large number of ED beds are occupied by admitted patients awaiting transfer to inpatient beds, physicians start to prioritize patients who will likely be discharged after treatment at ED, to avoid further blocking the ED. To do so, physicians first review details of patients waiting to be seen, including vital signs, demographics, early test results, etc., through the ED information system. This process takes time, but it helps predict the disposition of patients (i.e., admit or discharge) and make the prioritization decision to achieve better system performance. Note that the information collection in this example refers to the process of a physician reviewing medical information of patients and classifying them by their dispositions. It is *not* the triage process done by triage nurses upon a patient's arrival at the ED, which is undisputably necessary.

In this paper, to identify conditions under which spending additional time for classification and prioritization is beneficial, we analyze a single-server queueing system where customers can be from one of two types with the type of a customer determining the penalty incurred for each unit of time spent in the system. In this model, customers arrive to the system according to a stationary Poisson process and their type information is unknown upon arrival. The server has two options: either serve a customer directly without collecting any information on its type or *triage*, i.e., investigate the type information of customers and classify them into two classes accordingly, before serving them. (Our model allows for imperfect classification, i.e., the server may make mistakes in classifying customers.) Based on this classification, the server can prioritize them for service in accordance with the system's objective, i.e., minimizing the long-run average cost. We provide more specifics about our queueing model and discuss how we formulate the decision problem that the server faces as a Markov decision process in Sections 3 and 4.

This information/delay trade-off has not received much attention in queueing and operations literature except for papers by Sun et al. (2018) and Levi et al. (2019a,b), which we review in detail in Section 2 and which do not consider arrivals of customers. To the best of our knowledge, this paper is the first to study the information/delay trade-off by considering non-negligible triage times in a priority queue setting. On the operations management literature, our results provide useful insights to managers who consider triage for prioritizing certain groups of customers. For example, we obtain a simple expression for a lower bound on the mean triage time such that if the expected time to triage a customer is larger than this bound, then it is not worthwhile to triage at all and forego prioritization. In cases where triage is sufficiently fast and hence worth implementing, we show that instead of performing triage on all customers, it is better to triage incoming customers only when the number of unclassified customers is sufficiently large compared to the total number of customers in the system. Both our numerical and theoretical analysis show that arrival rate has an important effect on the triage decision, which differentiates our work from Sun et al. (2018) and

Levi et al. (2019a,b) since their models do not have arrivals. More specifically, through the comparison of two simple state-independent policies (no-triage and triage-all), we find that the information from triage is more beneficial *when the arrival rate is neither too small nor too large* and there is a good mix of type 1 and type 2 customers. Through extensive numerical studies, we found that as the traffic intensity increases, the sub-optimality of state-independent policies increases especially for cases where the performance of the no-triage policy is similar to that of triage-all policy. Hence, when traffic intensity is light, triage can be bypassed. When traffic intensity is mediocre to high, a more-complex state-dependent triage policy is needed if the two state-independent policies do not differ much in performance; otherwise, the best state-independent policy is fine.

The outline of the paper is as follows. We provide a review of the relevant literature in Section 2 and details of our mathematical model in Section 3. Our analytical results are presented in Sections 4 and 5. (Proofs of these results are deferred to the Appendix.) In Section 6, we investigate when and how much state-dependent policies are better than state-independent ones through a numerical study. We also test the robustness of our insights with respect to the preemption assumption. Section 7 concludes the paper with a discussion on the most important managerial insights derived from this work.

2. Literature Review

Priority queues have been studied extensively under the assumption that types of all customers are known perfectly. In particular, under linear waiting costs, an index policy, namely the $c\mu$ rule, has been shown to be optimal by Smith (1956), Cox and Smith (1961), and Kakalik and Little (1971). According to the $c\mu$ rule, a customer of type j has priority over a customer of type k if and only if $c_j\mu_j > c_k\mu_k$, where c_i and μ_i are the cost and service rates of a type i customer, respectively. A $c\mu$-type index policy is further shown to be optimal under various settings, see, e.g., Klimov (1974), Harrison (1975), Tcha and Pliska (1977), Pinedo (1983) and Budhiraja et al. (2014). For a comprehensive introduction on this topic, see Pinedo (2008). When the delay cost is convex, a generalized version of the $c\mu$ rule is shown to be asymptotically optimal, see Van Mieghem (1995) and Mandelbaum and Stolyar (2004). However, when the type identities of customers are not known perfectly, the service provider may prioritize the incorrect customers. Argon and Ziya (2009) consider this possibility and study the problem of priority assignment in a queueing system under the assumption that the type information of each customer is imperfect but this information is readily available. Saghafian et al. (2014) consider the possibility of misclassification in the context of patient prioritization in emergency departments but they do not focus on the question of whether to triage or not.

The studies discussed above all make the assumption that the type information (perfect or not) is readily available to the server upon a customer's arrival or at decision epochs. There are few studies that assume that the type information of customers are unknown but the server can choose to perform tests to obtain this information. Alizamir et al. (2013) consider a model where a single server classifies each arriving customer into one of two classes based on the results of a series of independent tests and the decision is to determine the number of tests to be performed to minimize the total cost of congestion and classification inaccuracy. Motivated by real-time internet traffic classification problems, Valdez-Vivas and Bambos (2013) consider a similar congestion-accuracy trade-off but the information retrieval process is modeled as stochastic arrivals of signals with different error rates according to some unknown arrival sequence. Another paper in this stream of literature is Wang et al. (2010). They study the operations of diagnostic service centers, where the service corresponds to performing diagnoses. The objective of this paper is to strike a balance between diagnostic accuracy, system congestion, and staffing costs by finding the appropriate capacity and service depth that are set before the system starts running, and hence, independently of changing congestion levels. All of these papers aim to balance costs associated with congestion and inaccuracy in classification and neither uses class information for prioritization of customers as in this work.

Our work is most relevant to three recent papers that study resource allocation with information acquisition for purposes of customer prioritization. Dobson and Sainathan (2011) compare two service systems: one system has sorters who collect information on a customer and then prioritize it before being served by processors, and the other system has only processors. In Dobson and Sainathan (2011), sorting and processing are done by different servers. In contrast, a single server is responsible for both actions in our model. Moreover, Dobson and Sainathan (2011) focus on static design questions while we investigate both the best static policies and optimal dynamic policies. Sun et al. (2018) investigate the patient triage and prioritization decisions in the aftermath of mass casualty events. Triage provides information on the urgency and service requirements, which are necessary for patient prioritization but at the cost of extra delay to patient treatment. Sun et al. (2018) show that the optimal triage decision can be characterized by a switching curve and provide a closed-form expression for this curve. Levi et al. (2019a,b) study a similar information/delay trade-off with multiple classes of customers and show that the structure of the optimal policy is again of threshold type. They develop near-optimal algorithms to solve the problem and quantify the value of information obtained through testing. Our work also studies the information/delay trade-off but in a queueing setting with external arrivals to the system in contrast to the clearing systems studied in Sun et al. (2018) and Levi et al. (2019a). The existence of an arrival process and the goal of minimizing the long-run average costs make the problem significantly more difficult to analyze. Specifically, the problems studied in Sun et al. (2018) and Levi et al. (2019a) are essentially optimal stopping problems under the optimal policy, which is no longer true for the problem under study in this paper, and hence, a fundamentally different approach is needed to characterize the optimal policy. In addition, we explicitly model the possibility of misclassification in our model, and our results also provide intuition on how the traffic intensity affects the service provider's decision on triage.

Finally, we should note that our work also shares some high-level similarities with other streams of work including studies on (i) two-tier service systems, see, e.g., Shumsky and Pinker (2003); (ii) cross-selling in a call center setting, see, e.g., Armony and Gurvich (2010); (iii) strategic behaviors under different levels of information, see, e.g., Guo and Zipkin (2007); (iv) learning information through service, see, e.g., Xu et al. (2015), Bimpikis and Markakis (2019), Shen et al. (2019). The models studied in these papers all have some level of information disclosure or learning, but the research focus is very different from ours.

3. Model Description

Consider a service system with a single server and two types of customers, namely, type 1 and type 2. Customers arrive at the system according to a Poisson process with rate $\lambda > 0$, and they wait in an infinite-capacity queue if the server is busy upon arrival. We refer to the new customers that have not been attended as "class 0" customers. Class 0 customers are a mixture of type 1 and type 2 customers but their true types are not known to the server. We assume that each class 0 customer belongs to type 1 with probability $p_1 \in (0, 1)$ and to type 2 with probability $p_2 \equiv 1 - p_1$.

The server can serve a class 0 customer without knowing its type, but s/he also has the option of spending some time on investigating the type of this customer before service, and classifying the customer as "class 1" or "class 2." The investigation time of a customer is exponentially distributed with mean $u > 0$, and is independent of the arrival process and the customer's type. In the rest of the paper, we use the term *triage* to refer to the process of investigating the type of a customer and classifying the customer based on that information. Define θ_i as the probability of classifying a type i customer as class $i \in \{1, 2\}$. Also, let q_i be the probability of classifying a class 0 customer as class $i \in \{1, 2\}$. Then, we have $q_1 = p_1\theta_1 + p_2(1 - \theta_2)$ and $q_2 = p_1(1 - \theta_1) + p_2\theta_2$.

Let $h_i \geq 0$ denote the (finite) per unit time cost that a type i customer incurs during its stay in the system for $i = 1, 2$. Let also r_i be the expected cost per unit time of a class i customer in the system for $i = 0, 1, 2$. It is then easy to show that $r_0 = p_1 h_1 + p_2 h_2$, $r_1 = (p_1\theta_1 h_1 + p_2(1 - \theta_2)h_2)/q_1$, and $r_2 = (p_1(1 - \theta_1)h_1 + p_2\theta_2 h_2)/q_2$. Note that $r_0 = q_1 r_1 + q_2 r_2$, i.e., r_0 is a convex combination of r_1 and r_2. Therefore, r_0 is either equal to both r_1 and r_2, or it is strictly between them.

We assume that service times of class i customers are independent and identically distributed exponential random variables with finite mean $\tau_i > 0$, $i = 0, 1, 2$. For analytical tractability, our theoretical results assume that a preemptive discipline is used, i.e., the server has the option of changing its action at any given time. This assumption may not be realistic. Indeed, most services are performed in a non-preemptive manner including in our motivating examples from healthcare. However, later in the paper, we study the non-preemptive case by means of a numerical study and show that our insights obtained for the preemptive discipline hold robustly under non-preemption.

Under the assumptions introduced above, at any point in time, there can be at most three classes of customers in the system with possibly different cost rates and service time distributions. The server can either serve one of these three classes of customers or can triage a class 0 customer into one of the two other classes. Our objective is to find policies that minimize the expected long-run average waiting costs.

In Section 4, we formulate this optimization problem as a Markov decision process (MDP) and provide characterizations of the optimal policy. Before we present these results, we would like to briefly discuss how the mathematical model defined above captures the main components of the motivating examples presented in Section 1. First of all, the existence of two types of customers in our model with differing holding cost rates captures the idea that the system controller values the wait of certain customers in the system more than the wait of others. The different service times mean that some customers require less efforts to complete, and switching the order of service may improve system efficiency. The positive triage time in our model represents that to determine the urgency (or "importance") of a customer, or to estimate the customer's service time, the server performs a brief preliminary review, which provides imperfect information on this customer's type and urgency but takes some time. More specifically, Tolun (2018) states that a geneticist may categorize a case to be urgent if s/he sees a suspicious test result during such a preliminary review, which takes some random amount of time and can lead to misclassification. Li et al. (2020) also note that a physician may spend some time to extract and review patients' information from ED information system, in order to estimate a patient's disposition and based on which the physician picks the next patient for treatment. Ibanez et al. (2018) find that processing tasks after triage is associated with superior performance, yet the time cost of reorganizing the queue may make triage inefficient and not worth it. All three examples are consistent with our modeling assumptions on the non-negligible triage times and the possibility of misclassification.

4. Optimal Dynamic Policies

To formulate the optimal control problem under study as a Markov decision process, let $\mathcal{S} \equiv \{\boldsymbol{x} \equiv (x_0, x_1, x_2) : x_i \geq 0,\ i = 0, 1, 2\}$ be the state space, where x_i is the number of customers in class i. To define the action space, note that at any given time the server can take one of the following four actions: $S0$ – serve a class 0 customer without triage (if $x_0 \geq 1$); $S1$ – serve a class 1 customer (if $x_1 \geq 1$); $S2$ – serve a class 2 customer (if $x_2 \geq 1$); and Tr – triage a class 0 customer (if $x_0 \geq 1$). One can easily show that idling is suboptimal when there is at least one customer in the system due to the preemption assumption. Hence, the action space is given by $\mathcal{A} \equiv \{S0, S1, S2, Tr\}$.

We next let

$$g(\pi, \boldsymbol{x}) \equiv \limsup_{t \to \infty} \frac{V_t(\pi, \boldsymbol{x})}{t},\ \forall \boldsymbol{x} \in \mathcal{S}, \tag{1}$$

be the expected long-run average cost, where $V_t(\pi, \boldsymbol{x})$ is the total expected cost up to time t starting from state $\boldsymbol{x}$ under policy π, which is a sequence of decision rules that map $\mathcal{S}$ to $\mathcal{A}$ and that specifies the action taken at any state and time. Then, the optimal expected long-run average cost is defined as

$$g^*(\boldsymbol{x}) \equiv \inf_{\pi} \; g(\pi, \boldsymbol{x}), \;\forall \boldsymbol{x} \in \mathcal{S}. \tag{2}$$

We first show the existence of an optimal stationary and deterministic policy π^* that satisfies (2). Let $\boldsymbol{e_i}$ be the ith row of a 3×3 identity matrix ($i = 1, 2, 3$), $\mathbf{0}$ be a 1×3 vector of all zeros, and $\boldsymbol{r} \equiv (r_0, r_1, r_2)^\intercal$ be a column vector of cost rates. We next apply *uniformization* with the uniformization constant $\phi \equiv \lambda + u^{-1} + \sum_{i=0}^{2} \tau_i^{-1}$ as in Lippman (1975). Without loss of generality, we can redefine the time unit so that $\phi = 1$, and thus, λ, u^{-1}, and τ_i^{-1} become respectively the probability that the next uniformized transition is an arrival, triage completion, and service completion of a class i customer, where $i = 0, 1, 2$. Let $v(\boldsymbol{x})$ be the relative cost function defined as the difference between the total expected cost starting from state $\boldsymbol{x}$ and a reference state (e.g., state $\mathbf{0}$). The long-run average cost optimality equations can be written as $v(\boldsymbol{x}) + g = Lv(\boldsymbol{x}), \;\forall \boldsymbol{x} \in \mathcal{S}$, where g is the optimal average cost per period of time after uniformization, and the operator L is defined as:

$$\begin{aligned} Lv(\boldsymbol{x}) = \lambda v(\boldsymbol{x} + \boldsymbol{e_1}) + \min \Big\{ & u^{-1}[q_1 v(\boldsymbol{x} - \boldsymbol{e_1} + \boldsymbol{e_2}) + q_2 v(\boldsymbol{x} - \boldsymbol{e_1} + \boldsymbol{e_3})] + (\tau_0^{-1} + \tau_1^{-1} + \tau_2^{-1}) v(\boldsymbol{x}), \\ & \tau_0^{-1} v(\boldsymbol{x} - \boldsymbol{e_1}) + (u^{-1} + \tau_1^{-1} + \tau_2^{-1}) v(\boldsymbol{x}), \; \tau_1^{-1} v(\boldsymbol{x} - \boldsymbol{e_2}) + (u^{-1} + \tau_0^{-1} + \tau_2^{-1}) v(\boldsymbol{x}), \\ & \tau_2^{-1} v(\boldsymbol{x} - \boldsymbol{e_3}) + (u^{-1} + \tau_0^{-1} + \tau_1^{-1}) v(\boldsymbol{x}) \Big\} + \boldsymbol{x} \cdot \boldsymbol{r}, \text{ if } \boldsymbol{x} \neq \mathbf{0}, \end{aligned} \tag{3}$$

$$Lv(\mathbf{0}) = \lambda v(\boldsymbol{e_1}) + (u^{-1} + \tau_0^{-1} + \tau_1^{-1} + \tau_2^{-1}) v(\mathbf{0}), \tag{4}$$

where $v(\cdot) : \mathcal{S} \longrightarrow \mathbb{R}$ and we assume that $v(\boldsymbol{x}) = \infty$ for $\boldsymbol{x} \notin \mathcal{S}$ for notational convenience. The first term in the right-hand side of (3) represents the deviation of the cost-to-go from the optimal average cost g associated with a new arrival and the last term represents the expected cost occurred until the next transition. The terms inside the minimization represent the deviation of cost-to-go from g associated with taking action $Tr, S0, S1, S2$, respectively. (Note that the last part of each of the four terms in the minimization represents a "dummy" transition due to uniformization, where the system state remains unchanged.) We are now ready to present Proposition 1. (The proofs of Proposition 1 and all other analytical results are provided in the Appendix.)

Proposition 1. *Assume that* $\lambda \tau_0 < 1$. *Then, there exist a function* $h(\cdot) : \mathcal{S} \longrightarrow \mathbb{R}$ *and a non-negative constant* g^* *that satisfy the average-cost optimality inequalities*

$$h(\boldsymbol{x}) + g^* \geq Lh(\boldsymbol{x}), \;\forall \boldsymbol{x} \in \mathcal{S}, \tag{5}$$

where $g^*(\boldsymbol{x}) = g^*$ *for all* $\boldsymbol{x} \in \mathcal{S}$. *Moreover, there exists an optimal stationary deterministic policy that achieves the minimum in* (5).

The condition $\lambda\tau_0 < 1$ in Proposition 1 ensures that the system is stable. Under this condition, Proposition 1 implies that there exists an optimal stationary deterministic policy with long-run average cost g^* that is independent of the initial state. The following theorem provides a partial characterization of this optimal policy π^* to the average-cost problem defined in (2). In general it is possible that there is more than one optimal action for any given state. If that is the case, for consistency, we choose the optimal action in the following order: $S1$, $S0$, Tr, and $S2$.

THEOREM 1. *Suppose that $\lambda\tau_0 < 1$ and $r_1/\tau_1 \geq r_0/\tau_0 \geq r_2/\tau_2$. Then, there exists an optimal stationary deterministic policy that solves the long-run average cost problem in* (2) *and takes the following form:*

(i) *When $x_1 \geq 1$, the optimal action is $S1$, i.e., always serve a class 1 customer when there is at least one.*

(ii) *The optimal action is $S2$ only when $x_0 = x_1 = 0$ and $x_2 \geq 1$, i.e., a class 2 customer should be served only when there is no other class of customers.*

(iii) *If $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$, $r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 \geq r_0\tau_0$, and*

$$u \geq \tilde{u} \equiv \frac{q_1(r_1\tau_0 - r_0\tau_1)}{r_0}, \tag{6}$$

then the optimal action is $S0$, i.e., a class 0 customer should be served without triage, in $\boldsymbol{x} \in \mathcal{S}$ with $x_0 \geq 1$ and $x_1 = 0$.

(iv) *If $\tau_0 = \tau_1 = \tau_2 = \tau$, $u < \tilde{u}$, and*

$$\lambda \leq \frac{1}{\tau + u}\left(1 - \frac{r_2}{(\tilde{u}/u - 1)r_0 + r_2}\right), \tag{7}$$

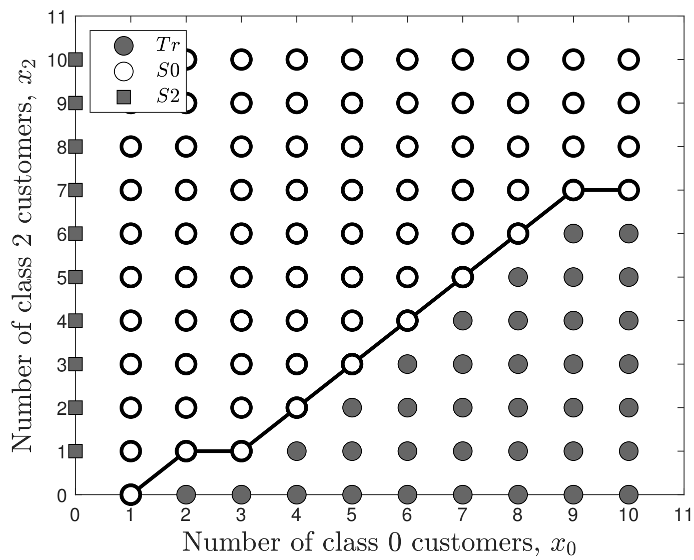
then for all $\boldsymbol{x} \in \mathcal{S}$ with $x_1 = 0$ and $x_0 \geq 1$, there exists a threshold $x_2^*(x_0)$ such that if $x_2 < x_2^*(x_0)$, the optimal action is Tr, i.e., triage a class 0 customer; otherwise, the optimal action is $S0$, i.e., a class 0 customer should be served without triage. Furthermore, $x_2^*(x_0)$ is a non-decreasing function of x_0.

In Theorem 1, we assume without loss of generality that $r_1/\tau_1 \geq r_2/\tau_2$, i.e., class 1 customers are more "important" than class 2 customers. Since class 0 customers are a mixture of customers of class 1 and class 2, it is reasonable to assume that r_0/τ_0 is between r_1/τ_1 and r_2/τ_2, i.e., class 0 customers are more important than class 2, but less important than class 1. Consistent with the classical $c\mu$ rule, parts (i) and (ii) of Theorem 1 show that at any given time and state, it is optimal to give class 1 customers the highest priority and class 2 customers the lowest priority, respectively. When there are no class 1 customers but at least one class 0 customer, then the server should attend a class 0 customer – either to triage the customer or to serve the customer directly without triage – in accordance with the structure identified in parts (iii) and (iv) of Theorem 1. More specifically, Theorem 1 (iii) shows that when triage takes a significant amount of time (so that the three conditions on u hold), the optimal policy simplifies to a state-independent policy that never performs triage on any class 0 customer and that directly serves them. The intuition is that the benefit of triage and prioritization diminishes when triage takes a significant amount of time, and hence, customers have to endure longer waits due to triage. Conditions in part (iii) give a precise description of what we mean by *triage taking a significant amount of time*. The first condition implies that it takes more time in expectation to first triage then serve a class 0 customer than to serve the customer without triage. Similarly, the second condition means that the expected cost charged while processing a class 0 customer is larger if it is triaged and served than if it is directly served. (Both conditions are automatically satisfied when $\tau_0 = \tau_1 = \tau_2$.) Additionally, by writing the right-hand side of condition (6) as $q_1\tau_0\tau_1(r_1/\tau_1 - r_0/\tau_0)/r_0$, we can see that skipping triage altogether would become optimal when there is not much difference in holding costs per unit of service between class 0 and class 1 customers, i.e., $r_1/\tau_1 - r_0/\tau_0$, is small.

On the other hand, when the opposite of (6) is true and the service times for all customers are identically distributed, and the arrival rate is sufficiently small (as in (7)), then the optimal policy is of threshold-type. In particular, when there are no class 1 customers in the system, then it is optimal to triage a class 0 customer if the number of class 2 customers is below a critical value, and serve a class 0 customer directly without triage when the number of class 2 customers in the system is sufficiently large. Figure 1 demonstrates this threshold structure by means of a numerical example. The intuition behind this threshold-type structure is that when there are many class 2 customers waiting for service, the value of type information obtained through triage could not compensate for the additional delay (as a result of triage) that the remaining customers will have to suffer. Hence, the optimal action is to skip triage. Furthermore, Theorem 1 (iv) shows that the threshold $x_2^*(x_0)$ is non-decreasing in x_0 when the arrival rate is bounded as in (7). The implication is that when there are more class 0 customers waiting in queue, the cost reduced by identifying and prioritizing an important customer is greater, which in turn means greater tolerance to the delay cost incurred by the less important customers in the system. It is important to note that although we need condition (7) to prove Theorem 1 (iv), the result may still hold under a weaker condition. In particular, note that (7) in fact is a sufficient condition for the system to be stable under all work-conserving and non-idling policies. More specifically, if (7) is satisfied, then $\lambda(\tau + u) < 1$, which ensures that the queueing system under every work-conserving and non-idling policy is stable. In a numerical study for stable queueing systems, despite trying a large number of scenarios, we were not able to find any case where $x_2^*(x_0)$ decreases with x_0 or is non-monotone. We also want to point out that our numerical experiments show that the threshold structure of the optimal policy may also hold for heterogeneous service times with distinct $\tau's$, although we can only prove it under the assumption of identical service times. Figure 1 is an illustration of the optimal policy when the mean service times are different.

Remark 1. Note that Theorem 1 has a similar structure to the main results in Sun et al. (2018). However, the proof is much more challenging than that for clearing systems as studied in Sun et al. (2018). The

Figure 1 **Visual depiction of the optimal policy when** $x_1 = 0, \lambda = 0.6, h_1 = 10, h_2 = 1, \tau_0 = 1, \tau_1 = 0.9, \tau_2 = 0.95, \theta_1 = \theta_2 = 0.9, u = 0.3,$ **and** $p_1 = 0.5.$ **Value-iteration algorithm with state space truncation at queue capacity** 100 **is used to find the optimal policy.**



optimality proof for the dynamic threshold policy in Sun et al. (2018) reduces to an optimal stopping problem because there are no future arrivals. When there is an arrival stream as in our motivating examples from healthcare, this trick can no longer be used. Hence, a whole new approach is needed, which requires intricate sample-path arguments and tedious algebra. In particular, to prove Proposition 1 and Theorem 1, we first prove similar existence and structural results for the corresponding discounted-cost problem that minimizes the total discounted cost over an infinite horizon; see Appendix A. We then use results from Section 7.2 of Sennott (1999) to extend these results to the average-cost problem by letting the discount factor go to zero; see Appendix B.

5. State-independent Policies

In Section 4, we showed that the optimal policy for the long-run average cost problem defined in (2) is state dependent and also provided an intuitive explanation as to why this is the case. However, in many applications, the system managers either triage all customers or do not triage at all, perhaps because of practicality or perception of fairness underlying these simple policies. Therefore, in this section, we explore the set of state-independent policies, i.e., policies that only use the state information as to whether x_i is zero or not for $i = 0, 1, 2$. Indeed, Theorem 1 implies that it is possible that under certain conditions (such as those in Theorem 1 (iii)), state-independent policies can be even optimal. We start by defining two such policies that are of particular interest.

- ***No-Triage Policy (NT):*** A non-idling policy, under which no customer goes through triage.

- ***Triage-Prioritize-Class-1 Policy (TP1):*** A non-idling policy, under which all customers are triaged, class 1 customers receive priority over all customers, and class 2 customers receive the lowest priority. Hence, if a customer is classified as class 1, it is served immediately, and class 2 customers are served only when there are no customers from another class in the system.

PROPOSITION 2. *When $\frac{r_1}{\tau_1} \geq \frac{r_2}{\tau_2}$, $\rho \equiv \lambda(u + q_1\tau_1 + q_2\tau_2) < 1$, $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$, $r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 \geq r_0\tau_0$, and $u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u) \geq \tau_0^2$, among all state-independent and deterministic policies, the policy that achieves the minimum long-run average cost is either TP1 or NT.*

Under some reasonable assumptions, Proposition 2 shows that it is sufficient to only consider policies *NT* and *TP1* to find an optimal policy that minimizes the expected long-run average cost within the set of all deterministic state-independent policies. The first condition $\frac{r_1}{\tau_1} \geq \frac{r_2}{\tau_2}$ means that class 1 customers are more important than class 2 customers, which is without loss of generality and consistent with our assumptions in Theorem 1. The stability of the system under policy *TP1* is guaranteed by $\rho < 1$. The next three conditions respectively imply that it takes more time for service in expectation, costs more during service, and results in a higher service time variance to first triage then serve a class 0 customer than to serve the customer directly without triage.

Besides being simple, easy-to-implement and potentially optimal, *NT* and *TP1* can also serve as benchmarks for any proposed dynamic policy. In the remainder of this section, we compare these two state-independent policies in terms of their expected long-run average costs. Denote the expected long-run average cost under policy π by c_π.

We first derive closed-form expressions for c_{NT} and c_{TP1}. For $\lambda\tau_0 < 1$, the system under *NT* is a stable M/M/1 queue with arrival rate λ and mean service time τ_0, and hence, from known results on M/M/1 queues (see, e.g., Section 7.3.1 in Kulkarni (2010)), we have

$$c_{NT} = \frac{\lambda\tau_0 r_0}{1 - \lambda\tau_0}. \tag{8}$$

It is not however straightforward to obtain an expression for c_{TP1}. Therefore, we state it as a proposition and provide a proof in Appendix C.

PROPOSITION 3. *Assume that $\rho < 1$. Then, we have*

$$c_{TP1} = \frac{\lambda^2(u^2 + q_1 u\tau_1 + q_1\tau_1^2)r_0}{1 - \lambda(u + q_1\tau_1)} + \lambda(r_0 u + q_1 r_1\tau_1) + \lambda q_2 r_2\left(u + \tau_2 + \frac{\rho[u + \tau_2 - \lambda u\tau_2 - \lambda q_1\tau_1(\tau_2 - \tau_1)]}{(1-\rho)[1 - \lambda(u + q_1\tau_1)]}\right). \tag{9}$$

Note that under *TP1*, there can be only two classes of customers in the queue in steady state: class 0 and class 2 customers. The first and second terms in (9) correspond to the long-run average holding cost of class

0 customers in queue and in service, respectively, while the third term corresponds to the long-run average cost of keeping class 2 customers in queue and service.

Our next result provides a comparison between *NT* and *TP1*, and hence, a characterization of the optimal policy within the set of deterministic state-independent policies when conditions of Proposition 2 hold.

THEOREM 2. *Assume that* $\rho < 1$ *and* $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$.

(i) *Suppose that* $r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 \geq r_0 \tau_0$. *There exist two thresholds* $\underline{\lambda}$ *and* $\bar{\lambda}$ *such that* $c_{NT} \leq c_{TP1}$ *if and only if* $\lambda \leq \underline{\lambda}$ *or* $\lambda \geq \bar{\lambda}$, *where* $0 < \underline{\lambda} \leq \bar{\lambda} < (u + q_1\tau_1 + q_2\tau_2)^{-1}$ *and all other system parameters are fixed.*

(ii) *Suppose that* $r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 \geq r_0 \tau_0$, $u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u) \geq \tau_0^2$, *and* $\tau_0 = \tau_2$. *There exist two thresholds* $\underline{p}_1$ *and* $\bar{p}_1$ *such that* $c_{NT} \leq c_{TP1}$ *if and only if* $p_1 \leq \underline{p}_1$ *or* $p_1 \geq \bar{p}_1$, *where* $0 \leq \underline{p}_1 \leq \bar{p}_1 \leq 1$ *and all other system parameters are fixed.*

(iii) *Suppose that* $\tau_0 = \tau_2$. $c_{NT} \leq c_{TP1}$ *if and only if* $\frac{r_2/\tau_2}{r_1/\tau_1} \geq \Theta_1$, *where*

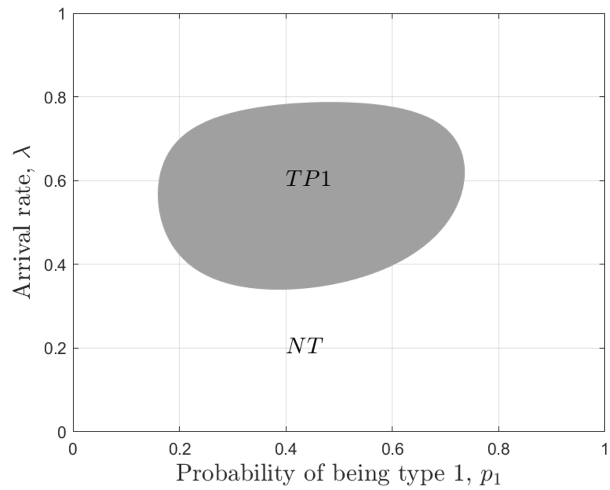
$$\Theta_1 \equiv \left[\frac{q_1\tau_1(q_2\tau_2)^{-1}(1-\rho)[\lambda q_2\tau_0\tau_1 - (1 - \lambda\tau_1 + \lambda^2\tau_0\tau_1)u + \tau_0 - \tau_1]}{(1-\lambda\tau_0)[\tau_0 + (1-\lambda\tau_0)u - \lambda q_1\tau_1(\tau_0 - \tau_1)] - (1-\rho)[1 - \lambda(u + q_1\tau_1)][\tau_0 - (1-\lambda\tau_0)u]} \right]^+$$

and $[x]^+ \equiv \max(0, x)$.

Most assumptions needed for Theorem 2 are the same as those for Proposition 2 and hence will not be discussed here. The only additional assumption needed is that we require $\tau_0 = \tau_2$ for parts (ii) and (iii) of Theorem 2, i.e., class 0 and class 2 customers have identically distributed service times but may be different from that of class 1. Although we have numerical evidence that this assumption is not necessary, we were not able to prove these results without it. However, note that this assumption may not be too restrictive especially for our motivating examples as we explain next. Under policy *TP1*, a class 0 customer will be served immediately if it is triaged as a class 1 whereas a class 0 customer triaged into class 2 will only get served when all class 0 customers are triaged and served. (This has also been shown to be optimal in Theorem 1 (i)&(ii).) It is then reasonable that the information from triage affects the service time of a class 1 customer because service of a class 1 immediately follows its triage. On the other hand, the information collected on a customer who is triaged into class 2 and served later should have little impact on its service, since the server may forget this information by the time this customer's service starts.

A visual illustration of parts (i) and (ii) of Theorem 2 is provided in Figure 2. Note that we plot this figure with heterogeneous service times, i.e., $\tau_i's$ are distinct. The structure in this figure is proved under the assumption $\tau_0 = \tau_2$, but we believe it holds under weaker conditions. Theorem 2 (i) shows the effect of the arrival rate on the choice between *NT* and *TP1*, and hence, in a way, on the value of the information obtained through triage. More specifically, it would be worthwhile *to triage all customers only when the traffic is*

Figure 2 **Best deterministic state-independent policy when** $h_1 = 10, h_2 = 1, \tau_0 = 1, \tau_1 = 0.9, \tau_2 = 0.95, u = 0.2$, **and** $\theta_1 = \theta_2 = 0.9$. **All conditions of Proposition 2 are satisfied in this plot.**



moderate; otherwise, i.e., when the traffic is light or heavy, it is better to skip triage for all customers. The intuition is that when the traffic is light, then the wait time any customer has to endure to get served is tolerable, and hence, the benefit from prioritization is not worth the extra cost incurred by triaging all customers. When the arrival rate is high, the wait times will be long and triaging *all* class 0 customers will make the wait times even longer, thus, it would be better to serve the customers without triage. From the expressions of c_{NT} and c_{TP1} in (8) and (9), we can numerically compute the interval $(\underline{\lambda}, \bar{\lambda})$, which tells us the range of λ where subjecting all customers to triage is better than skipping it altogether.

Part (ii) of Theorem 2 implies that *TP1* outperforms *NT* when there is a balanced mix of both types of customers, i.e., p_1 is neither too small nor too large. The intuition is that when p_1 is very small or large, either important customers are so rare that triage rarely ends up identifying them or they are so dominant that triage rarely helps eliminate the less important customers for immediate service.

In part (iii) of Theorem 2, it can be shown that $\Theta_1 > 0$ if and only if $u < (\lambda q_2 \tau_0 \tau_1 + \tau_0 - \tau_1)/(1 - \lambda \tau_1 + \lambda^2 \tau_0 \tau_1)$; see proof of Theorem 2 (iii). Hence, to triage all customers is better than to triage none when the mean triage time is sufficiently small and the importance level between the two classes is significantly different (i.e., $\frac{r_2/\tau_2}{r_1/\tau_1} < \Theta_1$, where $\Theta_1 < 1$ by Corollary 1 below).

We next use Theorem 1 (iii) and Theorem 2 (iii) to obtain Corollary 1, which provides guidance as to when triage should be considered. Figure 3 provides a pictorial description of this result.

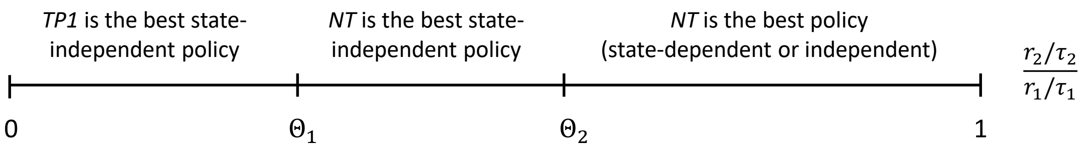
Corollary 1. *Suppose that* $\frac{r_1}{\tau_1} \geq \frac{r_0}{\tau_0} \geq \frac{r_2}{\tau_2}$, $\rho < 1$, $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$, $r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 \geq r_0 \tau_0$, $u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u) \geq \tau_0^2$, *and* $\tau_0 = \tau_2$. *Let* $\Theta_2 \equiv \left[\frac{q_1\tau_1(\tau_0 - q_1\tau_1 - u)}{q_2\tau_2(q_1\tau_1 + u)}\right]^+$, *where* $\Theta_1 \leq \Theta_2 < 1$ *for any* $u > 0$.

(i) If $\frac{r_2/\tau_2}{r_1/\tau_1} \geq \Theta_2$*, then there is no policy (state-dependent or -independent) that has a smaller long-run average cost than NT.*

(ii) If $\Theta_1 \leq \frac{r_2/\tau_2}{r_1/\tau_1} < \Theta_2$*, then there is no deterministic state-independent policy that has a smaller long-run average cost than NT.*

(iii) If $\frac{r_2/\tau_2}{r_1/\tau_1} < \Theta_1$*, then there is no deterministic state-independent policy that has a smaller long-run average cost than TP1.*

Figure 3 Optimality of *NT* and *TP1* as a function of $\frac{r_2/\tau_2}{r_1/\tau_1}$.



Corollary 1 and Figure 3 provide useful managerial insights into when triage should be ruled out and when it should be considered. First, when $u \geq \tau_0 - q_1\tau_1$, then $\Theta_2 = 0$, and hence by Corollary 1 triage should not be used at all. This simple condition has an intuitive explanation: The benefit of triage mainly comes from the prioritization of "important" customers, which are class 1 customers since their importance level r_1/τ_1 is the largest among all customers. However, triage comes at the cost of making all customers in the system wait for an extra u units of time on average. Taking both the cost and benefit of triage into account, $\frac{q_1 r_1}{u+q_1\tau_1}$ can be interpreted as the expected importance level of a customer who would be triaged as class 1. Given that $r_0 = q_1 r_1 + q_2 r_2 \geq q_1 r_1$, the condition $u + q_1\tau_1 > \tau_0$ implies that $\frac{r_0}{\tau_0} > \frac{q_1 r_1}{u+q_1\tau_1}$, i.e., a class 0 customer becomes less "important" after triage even if s/he would be triaged as class 1. Hence, there is no benefit in triage and *NT* is optimal.

Second, if triage is fast enough so that $\Theta_2 > 0$, then one should consider the heterogeneity of customer classes in terms of their differences in their "importance levels," measured by $r_i/\tau_i, i = 1, 2$, to decide whether to triage or not. More specifically, if $\frac{r_2/\tau_2}{r_1/\tau_1} \in [\Theta_2, 1)$, i.e., the two classes of customers do not differ significantly from one another, then triage should be ruled out completely; otherwise, some level of triage can be useful. If the management finds that triage can be useful but is not willing to implement a dynamic policy but wants to rather employ simple state-independent and deterministic policies, they should again rule out triage if $\frac{r_2/\tau_2}{r_1/\tau_1} \in [\Theta_1, \Theta_2)$. Finally, if $\frac{r_2/\tau_2}{r_1/\tau_1} < \Theta_1$, e.g., when triage is fast, arrival rate is mediocre, and/or there is a significant difference between the "importance levels" of the two classes, then to triage every customer is a better alternative to triage nobody – although there can be a better state-dependent triage policy in such a situation.

6. Numerical Study

In this section, we aim to investigate two main research questions through a numerical study. First, we have shown that the optimal policy could be a state-dependent policy that only triages incoming customers when the number of unclassified customers in the system is sufficiently large. However, we do not know when and how much these more complex policies are better than simpler state-independent policies such as the triage-all policy *TP1*. Second, our theoretical results are developed under the preemptive service discipline, and hence, we would like to test whether our insights hold under non-preemptive service.

6.1. Study Settings

When obtaining the performance for the optimal policy numerically, we retain the Markovian structure of the problem throughout our numerical experiments, and assume service and triage times are exponentially distributed. Without loss of generality, we set both the mean service time τ_2 and the cost incurred per unit time by a class 2 customer r_2 to one. We continue to assume that customers of class 1 are more important than that of class 0, which are more important than that of class 2, i.e., $r_1/\tau_1 \geq r_0/\tau_0 \geq r_2/\tau_2 = 1$. To examine the impact of the relative importance among different classes of customers, we consider three different values of r_1/τ_1, specifically $\{2, 5, 10\}$. We also set $\tau_1 = 0.9$ and choose τ_0 from $\{1, 1.05\}$, which means that information from triage reduces the mean service time for class 1 customers, but not necessarily for class 2 customers (see the discussion following Theorem 2 for why this would be a practical setting). Under assumptions $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$ and $r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 \geq r_0 \tau_0$, we know that the state-independent policy *NT* is optimal when $u \geq \tilde{u}$ by Theorem 1 (iii). Since we are interested in when and how much improvement an optimal dynamic policy can bring, in our experimental setup, we restrict the mean triage time u to be smaller than $\tilde{u}$. More specifically, we set $u = \hat{u} + \eta(\tilde{u} - \hat{u})$, where $\hat{u} \equiv \max\{\tau_0 - q_1\tau_1 - q_2\tau_2, \tau_0 - q_1 r_1 \tau_1/r_0 - q_2 r_2 \tau_2/r_0\}$ and $\eta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ (so that assumptions of Theorems 1 and 2 on u are satisfied and $u \leq \tilde{u}$). We consider five different values for q_1, i.e., the probability of classifying a customer as class 1, from the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. To examine the impact of different traffic levels, we choose λ so that $\rho = \lambda(u + q_1\tau_1 + q_2\tau_2)$ is in $\{0.3, 0.5, 0.7, 0.9\}$. Finally, we conduct all experiments under both preemptive and non-preemptive service disciplines. As a result, we consider a total of $2 \times 3 \times 5 \times 4 \times 5 \times 2 = 1200$ scenarios in this numerical study.

The long-run average costs of *NT* (with or without preemption) and *TP1* under preemption are computed by (8) and (9), respectively. To numerically compute the optimal long-run average cost for the underlying MDP formulation with an infinite state space, we needed to truncate the state space. More specifically, we applied the value-iteration algorithm to compute the optimal long-run average cost under the assumption that the system capacity is N, i.e., at any given time the total number of customers in the system $\sum_{i=0}^{2} x_i$ does not exceed N and a customer that arrives when the system is full, i.e., $\sum_{i=0}^{2} x_i = N$, is lost. The long-run average costs of the optimal policy and *TP1* under the non-preemptive discipline are computed under the same state space truncation. Clearly, such a truncation mechanism would result in only an approximation for the long-run average cost of each policy. To have a certain level of confidence in the accuracy of these approximate results, we varied the system capacity N from 60 to 120 in increments of 20 and obtained the long-run average cost of each policy for each scenario. We found that for all 1200 scenarios described above and the numerical precision we report in this section, the results were the same for $N = 100$ and larger. Hence, all results presented in this section are based on computations for $N = 100$.

6.2. Numerical Results

Tables 1 through 4 show the percentage increase in the expected long-run average cost by using the best of *NT* and *TP1*—either to skip triage altogether (i.e., *NT*) or to triage all incoming customers to give the highest priority to class 1 customers and the lowest priority to class 2 customers (i.e., *TP1*)—over the optimal policy under the preemptive service discipline for all 1200 scenarios described above. In all four tables, cells corresponding to scenarios where *NT* (*TP1*) is the best policy between *NT* and *TP1* are shaded (unshaded). Next, we discuss our observations from these tables on the impact of each of the system parameters and the robustness of the insights with respect to the preemption assumption.

6.2.1. Impact of the expected triage time (u) and the probability of being class 1 (q_1). We observe a large variation among scenarios with respect to the level of suboptimality of the best state-independent policy between *NT* and *TP1*. More specifically, when u is small and q_1 is large, or when u is large and q_1 is small, there is not much to be gained by using a dynamic policy over using a state-independent policy. In particular, for each fixed pair of ρ and r_1/τ_1 considered, *NT* is the better one between *NT* and *TP1* when η (and thus u) is large and q_1 is small (see the upper right corner of the (q_1, η) quadrant), and it does not perform much worse than the optimal dynamic policy. Similarly, *TP1* is better than *NT* when η (and thus u) is small and q_1 is large (see the lower left corner of the (q_1, η) quadrant) and performs similarly to the optimal policy. On the other hand, scenarios where the best of *NT* and *TP1* performs the worst lie close to the border between shaded and unshaded cells. A closer look at the parameters reveal that for these cells, $\frac{r_2/\tau_2}{r_1/\tau_1}$ is close to Θ_1, which means that neither *TP1* nor *NT* dominates the other significantly; see part (iii) of Theorem 2. Hence, the proximity of $\frac{r_2/\tau_2}{r_1/\tau_1}$ to Θ_1 can be used as a criterion to decide whether it is worthwhile to implement a more complex dynamic policy over a state-independent one.

6.2.2. Impact of the importance level (r_1/τ_1). From Tables 1 and 2, we observe that as r_1/τ_1 increases (i.e., as the "importance" of class 1 customers increases), the gap between the optimal dynamic policy and the best of *NT* and *TP1* becomes larger in general, except for some scenarios where η (thus the triage time) is small and q_1 represents a good mix of class 1 and class 2. Since r_2/τ_2 is a constant, the results imply that

Table 1 Percentage increase in the expected long-run average cost by using the best of *NT* and *TP1* over the optimal policy under preemptive service for $\tau_0 = 1,\ \tau_1 = 0.9,\ \tau_2 = 1,\ r_2 = 1$, and $\rho \in \{0.3, 0.5, 0.7, 0.9\}$. Policy *NT* (*TP1*) is the best between *NT* and *TP1* in the shaded (unshaded) cells.

	$\rho = 0.3$					$\rho = 0.5$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	1.54	0.88	0.36	0.06	0.00	1.69	1.52	0.75	0.22	0.01
$q_1 = 0.3$	2.83	2.04	0.86	0.14	0.00	2.64	3.85	1.92	0.54	0.01
$q_1 = 0.5$	2.33	2.53	1.13	0.19	0.00	1.95	3.62	2.73	0.87	0.03
$q_1 = 0.7$	1.31	1.92	1.21	0.23	0.00	1.01	1.56	2.37	1.35	0.09
$q_1 = 0.9$	0.41	0.44	0.49	0.56	0.12	0.31	0.33	0.36	0.39	0.47
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	4.62	2.96	1.10	0.15	0.00	4.67	5.29	2.40	0.59	0.01
$q_1 = 0.3$	4.37	5.23	1.91	0.23	0.00	4.10	9.78	4.66	1.04	0.01
$q_1 = 0.5$	2.19	5.38	2.09	0.28	0.00	1.98	5.10	5.60	1.33	0.01
$q_1 = 0.7$	0.80	2.77	1.93	0.31	0.00	0.63	1.33	3.11	1.73	0.03
$q_1 = 0.9$	0.19	0.21	0.27	0.49	0.00	0.14	0.16	0.18	0.21	0.35
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	6.34	5.12	1.77	0.20	0.00	6.52	9.56	4.08	0.85	0.00
$q_1 = 0.3$	4.16	7.32	2.52	0.27	0.00	3.60	9.93	6.60	1.31	0.00
$q_1 = 0.5$	1.75	6.80	2.58	0.32	0.00	1.30	4.64	7.30	1.65	0.01
$q_1 = 0.7$	0.44	2.84	2.25	0.34	0.00	0.35	0.95	3.01	1.95	0.03
$q_1 = 0.9$	0.10	0.11	0.43	0.46	0.00	0.07	0.08	0.09	0.11	0.23

	$\rho = 0.7$					$\rho = 0.9$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	2.21	2.03	1.06	0.40	0.04	4.76	2.31	1.15	0.48	0.09
$q_1 = 0.3$	3.27	5.70	2.92	1.13	0.11	11.37	6.48	3.45	1.47	0.29
$q_1 = 0.5$	2.17	5.14	4.70	1.88	0.25	10.84	10.66	5.79	2.61	0.75
$q_1 = 0.7$	0.83	1.77	3.34	3.31	1.04	4.67	11.67	8.38	4.12	1.60
$q_1 = 0.9$	0.20	0.21	0.25	0.32	0.51	0.99	1.29	2.13	3.63	4.31
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	6.79	7.38	3.62	1.16	0.06	15.19	8.54	4.33	1.61	0.19
$q_1 = 0.3$	6.09	13.72	7.97	2.36	0.08	28.21	21.89	10.85	3.86	0.33
$q_1 = 0.5$	2.65	6.40	10.58	3.47	0.15	15.45	31.62	15.90	5.95	0.61
$q_1 = 0.7$	0.62	1.54	3.02	4.78	0.43	4.50	11.14	18.48	8.67	1.62
$q_1 = 0.9$	0.09	0.11	0.12	0.17	0.30	0.60	0.39	1.14	2.03	3.77
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	9.66	14.13	6.59	1.85	0.05	31.71	17.43	8.57	2.88	0.21
$q_1 = 0.3$	5.59	12.74	12.28	3.29	0.06	27.53	38.44	18.39	6.04	0.33
$q_1 = 0.5$	1.82	4.74	8.98	4.68	0.14	11.80	24.01	24.81	9.07	0.71
$q_1 = 0.7$	0.32	0.87	1.86	3.89	0.44	2.61	6.52	10.84	12.42	1.90
$q_1 = 0.9$	0.05	0.06	0.06	0.08	0.12	0.05	0.86	0.33	0.70	1.76

when class 1 customers become more "important" than class 2 customers, then an optimal dynamic policy that makes state-dependent triage decisions brings larger benefit compared to state-independent policies. However, if the triage time is sufficiently short and there is a good mix of customers, then the triage-all policy (*TP1*) is nearly optimal and the optimality gap becomes smaller as customers become more different in their "importance" levels.

Table 2 Percentage increase in the expected long-run average cost by using the best of *NT* and *TP1* over the optimal policy under preemptive service for $\tau_0 = 1.05,\ \tau_1 = 0.9,\ \tau_2 = 1,\ r_2 = 1$, and $\rho \in \{0.3, 0.5, 0.7, 0.9\}$. Policy *NT* (*TP1*) is the best between *NT* and *TP1* in the shaded (unshaded) cells.

	$\rho = 0.3$					$\rho = 0.5$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	1.68	1.25	0.90	0.61	0.34	2.77	2.18	1.70	1.29	0.97
$q_1 = 0.3$	3.75	1.86	0.89	0.17	0.00	3.33	4.29	2.29	0.84	0.07
$q_1 = 0.5$	3.23	2.55	1.20	0.23	0.00	2.52	3.62	3.38	1.25	0.11
$q_1 = 0.7$	1.90	2.26	1.35	0.45	0.00	1.45	1.73	2.26	2.50	0.68
$q_1 = 0.9$	0.60	0.63	0.65	0.68	0.74	0.45	0.47	0.49	0.51	0.54
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	4.86	2.65	1.09	0.17	0.00	5.06	4.84	2.37	0.62	0.01
$q_1 = 0.3$	4.53	5.01	1.93	0.25	0.00	4.29	8.71	4.68	1.09	0.01
$q_1 = 0.5$	2.45	5.26	2.14	0.31	0.00	2.09	4.19	5.78	1.42	0.02
$q_1 = 0.7$	1.03	2.14	2.06	0.35	0.00	0.82	1.23	2.17	2.15	0.06
$q_1 = 0.9$	0.26	0.28	0.30	0.37	0.04	0.20	0.22	0.23	0.25	0.28
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	6.71	4.81	1.76	0.22	0.00	6.49	9.11	4.02	0.89	0.01
$q_1 = 0.3$	3.88	7.11	2.53	0.29	0.00	3.57	8.86	6.67	1.36	0.00
$q_1 = 0.5$	1.57	6.09	2.64	0.34	0.00	1.32	3.69	7.49	1.74	0.01
$q_1 = 0.7$	0.57	2.12	2.37	0.38	0.00	0.46	0.71	2.07	2.34	0.05
$q_1 = 0.9$	0.14	0.15	0.16	0.58	0.02	0.10	0.11	0.12	0.13	0.15

	$\rho = 0.7$					$\rho = 0.9$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	1.96	2.46	2.09	1.62	1.29	4.63	3.33	2.38	1.69	1.19
$q_1 = 0.3$	2.91	5.59	3.61	1.91	0.76	10.46	6.08	3.54	1.92	0.97
$q_1 = 0.5$	2.10	4.12	5.67	2.94	1.11	8.44	10.59	6.13	3.21	1.49
$q_1 = 0.7$	1.01	1.57	2.61	4.17	2.41	3.67	9.07	9.34	5.21	2.58
$q_1 = 0.9$	0.29	0.30	0.31	0.35	0.42	0.15	1.12	1.60	2.63	4.38
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	5.81	7.13	3.55	1.33	0.12	13.44	7.83	4.22	1.83	0.35
$q_1 = 0.3$	5.34	11.96	7.96	2.61	0.12	24.39	21.44	10.84	4.05	0.44
$q_1 = 0.5$	2.38	5.57	9.70	3.89	0.23	13.22	29.11	16.39	6.45	0.85
$q_1 = 0.7$	0.67	1.30	2.44	4.27	0.81	3.63	9.51	16.32	9.77	2.60
$q_1 = 0.9$	0.13	0.14	0.16	0.18	0.23	0.08	0.68	0.91	1.54	3.00
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	8.65	13.48	6.46	1.91	0.07	29.82	16.68	8.26	2.99	0.28
$q_1 = 0.3$	4.84	11.03	12.37	3.47	0.08	24.34	38.15	18.35	6.31	0.40
$q_1 = 0.5$	1.57	4.00	7.56	5.08	0.19	10.11	21.59	25.44	9.51	0.88
$q_1 = 0.7$	0.37	0.70	1.41	2.73	0.70	2.06	5.49	9.51	13.72	2.84
$q_1 = 0.9$	0.07	0.08	0.08	0.09	0.10	0.04	0.08	0.20	0.47	1.10

6.2.3. Impact of the traffic intensity (ρ), the service times, and the preemptive assumption. We observe that as the traffic intensity increases, the gap between the optimal dynamic policy and the best of *NT* and *TP1* becomes larger in general, especially for the scenarios that lie close to the boarder between the shaded and unshaded cell, i.e., when $\frac{r_2/\tau_2}{r_1/\tau_1}$ is close to Θ_1. We have fixed the mean service times of classes 1 and 2 customers but vary that of class 0 customers from $\tau_0 = 1$ (results in Table 1) to $\tau_0 = 1.05$ (results in

Table 3 **Percentage increase in the expected long-run average cost by using the best of *NT* and *TP1* over the optimal policy under non-preemptive service for $\tau_0 = 1.0,\ \tau_1 = 0.9,\ \tau_2 = 1,\ r_2 = 1$, and $\rho \in \{0.3, 0.5, 0.7, 0.9\}$. Policy *NT* (*TP1*) is the best between *NT* and *TP1* in the shaded (unshaded) cells.**

	$\rho = 0.3$					$\rho = 0.5$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	0.42	0.21	0.06	0.01	0.00	0.76	0.58	0.21	0.04	0.00
$q_1 = 0.3$	0.82	0.48	0.13	0.01	0.00	0.90	1.43	0.51	0.09	0.00
$q_1 = 0.5$	0.30	0.60	0.19	0.02	0.00	0.35	1.99	0.76	0.14	0.00
$q_1 = 0.7$	0.03	0.59	0.23	0.03	0.00	0.04	0.48	1.01	0.23	0.00
$q_1 = 0.9$	0.00	0.00	0.01	0.09	0.00	0.00	0.00	0.01	0.05	0.15
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	1.41	0.66	0.16	0.01	0.00	3.19	1.87	0.61	0.09	0.00
$q_1 = 0.3$	2.35	1.13	0.25	0.02	0.00	3.21	3.54	1.11	0.15	0.00
$q_1 = 0.5$	1.83	1.19	0.30	0.02	0.00	1.52	4.10	1.37	0.20	0.00
$q_1 = 0.7$	0.29	1.01	0.31	0.03	0.00	0.10	2.81	1.54	0.26	0.00
$q_1 = 0.9$	0.00	0.01	0.29	0.05	0.00	0.00	0.00	0.04	0.53	0.01
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	2.36	1.08	0.23	0.02	0.00	4.97	3.16	0.95	0.12	0.00
$q_1 = 0.3$	3.17	1.51	0.32	0.02	0.00	3.92	4.93	1.46	0.17	0.00
$q_1 = 0.5$	2.40	1.46	0.35	0.03	0.00	1.88	5.20	1.71	0.23	0.00
$q_1 = 0.7$	0.61	1.17	0.35	0.03	0.00	0.07	3.55	1.78	0.30	0.00
$q_1 = 0.9$	0.00	0.11	0.30	0.05	0.00	0.00	0.00	0.27	0.50	0.01

	$\rho = 0.7$					$\rho = 0.9$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	1.57	1.09	0.45	0.11	0.00	3.10	1.60	0.71	3.90	1.69
$q_1 = 0.3$	2.00	2.91	1.18	0.27	0.00	8.94	4.57	1.99	0.61	0.02
$q_1 = 0.5$	1.11	4.19	1.83	0.45	0.01	9.99	7.47	3.30	1.03	0.04
$q_1 = 0.7$	0.20	1.09	2.67	0.77	0.02	3.90	10.95	5.03	1.63	0.11
$q_1 = 0.9$	0.00	0.00	0.03	0.13	0.40	0.09	0.62	1.71	3.47	2.18
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	5.75	3.66	1.39	0.26	0.00	11.85	5.85	2.40	0.59	0.01
$q_1 = 0.3$	5.19	7.73	2.82	0.51	0.00	26.27	13.82	5.47	1.22	0.01
$q_1 = 0.5$	2.19	7.85	3.85	0.77	0.00	14.11	20.12	8.27	1.98	0.03
$q_1 = 0.7$	0.28	2.43	4.72	1.14	0.01	4.08	11.10	11.47	3.43	0.10
$q_1 = 0.9$	0.00	0.00	0.04	0.19	0.23	0.05	0.37	1.07	2.35	1.10
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	8.38	6.47	2.28	0.38	0.00	25.13	10.87	4.17	0.88	0.01
$q_1 = 0.3$	5.48	11.42	4.01	0.65	0.00	24.77	22.21	8.40	1.69	0.01
$q_1 = 0.5$	2.13	7.90	5.10	0.97	0.00	10.73	24.06	12.08	2.82	0.03
$q_1 = 0.7$	0.17	2.67	5.83	1.40	0.01	2.46	7.24	13.42	4.66	0.11
$q_1 = 0.9$	0.00	0.00	0.02	0.46	0.18	0.01	0.13	0.44	1.10	1.21

Table 2). We observe that the gap between the optimal policy and the best of *NT* and *TP1* becomes smaller when the service times of customers are reduced more by triage. Finally, Tables 3 and 4 show that all insights obtained from our numerical analysis with preemption continue to hold when the preemption assumption is relaxed. Interestingly, we find that the best of *NT* and *TP1* performs closer to the optimal policy under non-preemptive service discipline.

Table 4 **Percentage increase in the expected long-run average cost by using the best of *NT* and *TP1* over the optimal policy under non-preemptive service for $\tau_0 = 1.05,\ \tau_1 = 0.9,\ \tau_2 = 1,\ r_2 = 1$, and $\rho \in \{0.3, 0.5, 0.7, 0.9\}$. Policy *NT* (*TP1*) is the best between *NT* and *TP1* in the shaded (unshaded) cells.**

	$\rho = 0.3$					$\rho = 0.5$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	0.01	0.05	0.12	0.18	0.00	0.04	0.13	0.28	0.52	0.56
$q_1 = 0.3$	0.10	0.45	0.17	0.02	0.00	0.21	0.98	0.62	0.14	0.00
$q_1 = 0.5$	0.05	0.60	0.23	0.03	0.00	0.11	0.69	0.94	0.21	0.00
$q_1 = 0.7$	0.00	0.09	0.29	0.05	0.00	0.01	0.13	0.49	0.40	0.01
$q_1 = 0.9$	0.00	0.00	0.00	0.01	0.04	0.00	0.00	0.00	0.00	0.02
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	0.79	0.62	0.17	0.02	0.00	1.00	1.79	0.61	0.10	0.00
$q_1 = 0.3$	1.37	1.12	0.29	0.02	0.00	1.14	3.58	1.19	0.17	0.00
$q_1 = 0.5$	0.44	1.20	0.35	0.03	0.00	0.35	4.22	1.55	0.23	0.00
$q_1 = 0.7$	0.01	1.04	0.37	0.04	0.00	0.01	1.03	1.78	0.36	0.00
$q_1 = 0.9$	0.00	0.00	0.01	0.10	0.00	0.00	0.00	0.00	0.03	0.10
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	2.22	1.06	0.25	0.02	0.00	2.69	3.15	0.97	0.13	0.00
$q_1 = 0.3$	2.29	1.51	0.36	0.02	0.00	1.84	5.04	1.58	0.19	0.00
$q_1 = 0.5$	0.97	1.48	0.40	0.03	0.00	0.29	5.36	1.89	0.26	0.00
$q_1 = 0.7$	0.00	1.21	0.40	0.03	0.00	0.00	1.78	2.05	0.38	0.00
$q_1 = 0.9$	0.00	0.00	0.38	0.08	0.00	0.00	0.00	0.00	0.09	0.05

	$\rho = 0.7$					$\rho = 0.9$				
η	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$r_1/\tau_1 = 2$										
$q_1 = 0.1$	0.21	0.59	1.10	1.23	0.88	2.43	1.69	1.20	0.85	0.60
$q_1 = 0.3$	0.90	3.06	1.31	0.37	0.01	8.37	4.47	1.99	0.62	0.39
$q_1 = 0.5$	0.50	2.18	2.16	0.63	0.02	7.33	7.84	3.60	1.13	0.22
$q_1 = 0.7$	0.06	0.53	1.54	1.33	0.10	2.67	8.40	6.33	2.56	0.33
$q_1 = 0.9$	0.00	0.00	0.01	0.03	0.12	0.03	0.32	1.03	2.28	4.27
$r_1/\tau_1 = 5$										
$q_1 = 0.1$	3.07	3.48	1.37	0.31	0.00	10.50	5.26	2.38	0.62	0.02
$q_1 = 0.3$	2.93	7.85	2.94	0.58	0.00	22.13	13.89	5.59	1.33	0.02
$q_1 = 0.5$	1.09	5.40	4.18	0.88	0.01	11.71	20.44	8.63	2.23	0.05
$q_1 = 0.7$	0.10	0.86	3.74	1.44	0.02	3.08	9.03	12.35	3.82	0.17
$q_1 = 0.9$	0.00	0.00	0.01	0.04	0.22	0.02	0.20	0.68	1.62	3.28
$r_1/\tau_1 = 10$										
$q_1 = 0.1$	5.50	6.40	2.32	0.43	0.00	21.99	10.58	4.20	0.95	0.01
$q_1 = 0.3$	3.12	11.69	4.21	0.73	0.00	21.23	22.59	8.77	1.89	0.02
$q_1 = 0.5$	0.81	5.64	5.53	1.11	0.00	8.72	20.79	12.74	3.15	0.04
$q_1 = 0.7$	0.04	0.85	4.17	1.72	0.02	1.74	5.48	10.97	5.33	0.19
$q_1 = 0.9$	0.00	0.00	0.00	0.01	0.15	0.00	0.05	0.23	0.65	1.71

In summary, the system manager should consider implementing a state-dependent triage policy when $\frac{r_2/\tau_2}{r_1/\tau_1}$ is close to Θ_1 (e.g., when q_1 and u are neither too large nor too small), when the heterogeneity in the importance levels of customers is large, and/or when the traffic intensity is high. The gap between the optimal policy and the best of *NT* and *TP1* can be over 30% in some scenarios.

7. Conclusion

In this paper, we studied a fundamental question many service systems with heterogeneous customers and arrivals face: When is it worthwhile to spend additional time on customers to triage them before service to improve the priority order of customers for processing? To find an answer to this question, we analyzed a stylized, single-server queueing model with two types of customers both analytically and numerically. This analysis resulted in several useful managerial insights. First, we found that implementing a triage policy for purposes of prioritization should not be considered if triage takes a significantly large amount of time in comparison to the actual service. This result is not surprising but we were able to derive a simple condition to decide whether triage is too long to implement: If $u + q_1\tau_1 > \tau_0$, i.e., the expected time it takes to triage a class 0 customer and serve her immediately it is classified as class 1 and skip it if it is of class 2 is larger than the expected time it takes to serve a customer without triage, then triage should be ruled out. If we find that triage is not that long, then we show that one needs to also take into account the heterogeneity of customers in the population in terms of their perceived value to the system and their service times to decide whether triage should be implemented.

By means of a Markov decision process formulation, we show that when the service times of all customers are identical and it is worthwhile to triage, the optimal triage policy is dependent on the number of customers in the system. More specifically, the optimal policy does not in general triage all arriving customers but only when the number of unclassified customers is sufficiently large in comparison with the number of other customers in the system. In other words, if the number of unclassified customers is smaller than a certain threshold, then it is better to directly serve these customers right away instead of classifying them by triage.

The threshold structure of the optimal policy is similar to that in Sun et al. (2018), which shows the robustness of the result and the insights from them. On the other hand, both our numerical and theoretical analysis show that arrival rate (and thus traffic intensity) has an important effect on the triage decision, which differentiates our work from Sun et al. (2018) and Levi et al. (2019a,b) since their models do not have arrivals. More specifically, through the comparison of two state-independent policies (no-triage and triage-all), we find that the information from triage is more beneficial *when the arrival rate is neither too small nor too large* and there is a good mix of type 1 and type 2 customers. Through extensive numerical studies, we found that as the traffic intensity increases, the sub-optimality of state-independent policies increases especially for cases where the performance of the no-triage policy is similar to that of triage-all policy. Hence, when traffic intensity is light, triage can be bypassed. When traffic intensity is mediocre to high, a more-complex state-dependent triage policy is needed if the two state-independent policies do not differ much in performance; otherwise, the best state-independent policy is fine.

Although the optimal triage policy should be state dependent in general, the comparison of the triage-all policy and the no-triage policy provides insights for managements that see value in a triage policy but do not want to employ a complex state-dependent one. This analysis resulted in an explicit necessary and sufficient condition to determine whether triage-all is better than no-triage. The condition depends on the relative "importance" of class 2 customers over class 1, i.e., $\frac{r_2/\tau_2}{r_1/\tau_2}$, and a complex term Θ_1 that depends on the arrival rate, percentage of important customers in the population, triage and service rates, and misclassification probabilities. From our numerical experiments, we observed that this condition can also inform when to prefer dynamic triage policies. In particular, we found that when $\frac{r_2/\tau_2}{r_1/\tau_2}$ is close to Θ_1, a policy that performs triage dynamically depending on the system state would perform significantly better than the best policy between no-triage and triage-all, especially when the traffic is heavy.

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Appendices

We provide proofs of our theoretical results in the appendices. In particular, in Appendix A, we provide the MDP formulation of our optimal control problem under the objective of minimizing the total discounted cost over an infinite horizon and we prove certain structural results for its optimal solution. In Appendix B, we extend the structural properties of the value functions for the infinite-horizon total discounted cost problem to the long-run average cost problem to prove Proposition 1 and Theorem 1. Finally, the proofs of analytical results presented in Section 5 are provided in Appendix C.

Appendix A. Discounted Cost Problem

Let $X_i^\pi(t)$ be the number of class i customers ($i = 0, 1, 2$) in the system at time $t \geq 0$ under policy π. The infinite-horizon discounted-cost problem is to find a policy π that minimizes

$$V^{\pi}(\boldsymbol{x}) = E\left[\int_0^{\infty} e^{-\alpha t}\sum_{i=0}^{2} r_i X_i^{\pi}(t)dt\Big|\boldsymbol{x}\right], \tag{10}$$

where $\boldsymbol{x}$ is the initial system state and $\alpha > 0$ is the continuous-time discount rate. We next apply *uniformization* with the uniformization constant $\phi = \lambda + u^{-1} + \sum_{i=0}^{2}\tau_i^{-1} + \alpha$ as in Lippman (1975). Without loss of generality, we can redefine the time unit so that $\phi = 1$. Then, the T-period total discounted cost under policy π is

$$V^{\pi}(\boldsymbol{x}, T) = E\left[\sum_{t=1}^{T}\gamma^t\sum_{i=0}^{2} r_i X_i^{\pi}(t)\Big|\boldsymbol{x}\right], \tag{11}$$

where the discounting factor is $\gamma \equiv \frac{\lambda+u^{-1}+\tau_0^{-1}+\tau_1^{-1}+\tau_2^{-1}}{\alpha+\lambda+u^{-1}+\tau_0^{-1}+\tau_1^{-1}+\tau_2^{-1}} = 1-\alpha$; see Figure 11.5.3 in Puterman (2005). The uniformized discrete-time version of the decision problem in (10) is to find a policy π that minimizes

$$V^{\pi}(\boldsymbol{x}) = \lim_{T\to\infty} V^{\pi}(\boldsymbol{x}, T). \tag{12}$$

Let $v(\boldsymbol{x}) = \inf_{\pi} V^{\pi}(\boldsymbol{x})$ be the minimum total discounted cost starting from state $\boldsymbol{x}$. The optimality equations for the total discounted cost problem defined in (10) can be written as $v(\boldsymbol{x}) = Lv(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \mathcal{S}$, where L is defined in (3) and (4). Now, we are ready to first show the existence of an optimal policy that minimizes the total discounted cost over an infinite horizon.

Proposition 4. *There exists an optimal stationary deterministic policy that solves the discounted-cost problem* (10)*, and the optimality equation $v(\boldsymbol{x}) = Lv(\boldsymbol{x})$, for $\boldsymbol{x} \in \mathcal{S}$, has a unique and finite solution $v(\cdot)$ that corresponds to the total discounted cost under this optimal policy.*

Proof: We prove Proposition 4 by checking the conditions of Theorem 11.5.3 in Puterman (2005). It is obvious that the state space $\mathcal{S}$ is countable and the action space at state $\boldsymbol{x}$, denoted by $\mathcal{A}_{\boldsymbol{x}}$, is finite for each $\boldsymbol{x} \in \mathcal{S}$. If we let $c \equiv \lambda + u^{-1} + \sum_{i=0}^{2}\tau_i^{-1} < \infty$, then Assumption 11.5.1 in Puterman (2005) holds because $c \geq \max_{\boldsymbol{x},a}\{\beta(\boldsymbol{x}, a)\}$ and $1 - q(\boldsymbol{x}|\boldsymbol{x}, a) < 1$, where $\beta(\boldsymbol{x}, a)$ is the transition rate when action a is chosen in state $\boldsymbol{x}$ in the underlying continuous-time MDP and $q(\boldsymbol{x'}|\boldsymbol{x}, a)$ denotes the transition probability from state $\boldsymbol{x}$ to $\boldsymbol{x'}$ when action a is chosen for the embedded chain underlying the continuous-time MDP. Next, we show that there exists a positive real-valued function $w(\boldsymbol{x})$ satisfying $\min_{\boldsymbol{x}\in\mathcal{S}} w(\boldsymbol{x}) > 0$ such that Assumptions 6.10.1 and 6.10.2 in Puterman (2005) hold. Let $w(\boldsymbol{x}) = \max\{1, x_0 + x_1 + x_2\}$ for $\boldsymbol{x} \in \mathcal{S}$. Then, $\min_{\boldsymbol{x}\in\mathcal{S}} w(\boldsymbol{x}) = 1 > 0$. Let $r(\boldsymbol{x}, a)$ be the expected immediate reward gained when action a is taken in state $\boldsymbol{x}$, then we have $r(\boldsymbol{x}, a) \equiv -r_0x_0 - r_1x_1 - r_2x_2$, and for all $\boldsymbol{x} \in \mathcal{S}$,

$$\max_{a\in\mathcal{A}_{\boldsymbol{x}}} |r(\boldsymbol{x}, a)| = |-r_0x_0 - r_1x_1 - r_2x_2| \leq \max\{r_0, r_1, r_2\}(x_0 + x_1 + x_2) \leq \max\{r_0, r_1, r_2\}w(\boldsymbol{x}).$$

Hence, Assumption 6.10.1 is satisfied. For $\boldsymbol{j} = (j_0, j_1, j_2)$ and $\boldsymbol{x} \in \mathcal{S}$,

$$\sum_{\boldsymbol{j}\in\mathcal{S}} p(\boldsymbol{j}|\boldsymbol{x}, a)w(\boldsymbol{j}) = \sum_{\boldsymbol{j}\in\mathcal{S}} p(\boldsymbol{j}|\boldsymbol{x}, a)\max\{1, j_0 + j_1 + j_2\} \leq \sum_{\boldsymbol{j}\in\mathcal{S}} p(\boldsymbol{j}|\boldsymbol{x}, a)\max\{1, x_0 + x_1 + x_2 + 1\},$$

where $p(\boldsymbol{j}|\boldsymbol{x},a)$ is the transition probability to state $\boldsymbol{j}$ when action a is taken in state $\boldsymbol{x}$ for the discrete-time MDP after uniformization, and the inequality holds because every period the number of customers in the system can change by at most one. Thus,

$$\sum_{\boldsymbol{j}\in\mathcal{S}} p(\boldsymbol{j}|\boldsymbol{x},a)w(\boldsymbol{j}) \le \max\{1, x_0+x_1+x_2+1\} = 1+x_0+x_1+x_2 \le 2w(\boldsymbol{x}),\ \forall a\in\mathcal{A}_{\boldsymbol{x}} \text{ and } \boldsymbol{x}\in\mathcal{S}.$$

Hence, part (a) of Assumption 6.10.2 is satisfied. For any policy π, integer $J\ge 1$ and $0\le c<1$, we have

$$c^J\sum_{\boldsymbol{j}\in\mathcal{S}} P_\pi^J(\boldsymbol{j}|\boldsymbol{x})w(\boldsymbol{j}) = c^J\sum_{\boldsymbol{j}\in\mathcal{S}} P_\pi^J(\boldsymbol{j}|\boldsymbol{x})\max\{1, j_0+j_1+j_2\} \le c^J\sum_{\boldsymbol{j}\in\mathcal{S}} P_\pi^J(\boldsymbol{j}|\boldsymbol{x})\max\{1, x_0+x_1+x_2+J\}$$
$$= c^J(x_0+x_1+x_2+J) \le c^J(J+1)w(\boldsymbol{x}),\ \forall\, \boldsymbol{x}\in\mathcal{S},$$

where $P_\pi^J(\boldsymbol{j}|\boldsymbol{x})$ is the $(\boldsymbol{x},\boldsymbol{j})$th component of the J-step transition probability matrix under policy π and the first inequality holds because the number of customers in the system can change by at most one every period. As long as J is sufficiently large, we know that $0\le c^J(J+1)<1$. Let $a=c^J(J+1)$, we have $c^J\sum_{\boldsymbol{j}\in\mathcal{S}} P_\pi^J(\boldsymbol{j}|\boldsymbol{x})w(\boldsymbol{j}) \le aw(\boldsymbol{x})$, and part (b) of Assumption 6.10.2 is satisfied. □

The following theorem provides a partial characterization of the optimal policy to problem (10). Here, we denote the action taken at state $\boldsymbol{x}$ under the optimal policy by $a^*(\boldsymbol{x})$.

THEOREM 3. *Assume that $r_1/\tau_1 \ge r_0/\tau_0 \ge r_2/\tau_2$. There exists an optimal stationary deterministic policy that solves the discounted-cost problem defined in* (10) *and takes the following form:*

(i) *If $x_1\ge 1$, then $a^*(\boldsymbol{x}) = S1$, i.e., a class 1 customer should be served.*

(ii) *$a^*(\boldsymbol{x}) = S2$, i.e., a class 2 customer should be served, only when $x_0=x_1=0$.*

(iii) *Suppose that $q_1r_1\tau_1/(1+\alpha\tau_1)+q_2r_2\tau_2/(1+\alpha\tau_2)+r_0u \ge r_0\tau_0$ and $q_1\tau_1/(1+\alpha\tau_1)+q_2\tau_2/(1+\alpha\tau_2)+u \ge \tau_0$. When $x_0=1$ and $x_1=0$, $a^*(\boldsymbol{x}) = S0$, i.e., the class 0 customer should be directly served.*

(iv) *Suppose that $q_1r_1\tau_1/(1+\alpha\tau_1)+q_2r_2\tau_2/(1+\alpha\tau_2)+r_0u \ge r_0\tau_0$ and $q_1\tau_1/(1+\alpha\tau_1)+q_2\tau_2/(1+\alpha\tau_2)+u \ge \tau_0$. If*

$$u \ge \tilde{u}(\alpha) \equiv q_1(r_1\tau_0 - r_0\tau_1)/[(1+\alpha\tau_1)r_0], \tag{13}$$

then $a^*(\boldsymbol{x}) = S0$ for all $\boldsymbol{x}\in\mathcal{S}$ with $x_0\ge 2$ and $x_1=0$, i.e., a class 0 customer should be directly served.

(v) *Suppose that $\tau_0=\tau_1=\tau_2=\tau$, $\alpha\tau^2/(1+\alpha\tau) < u < \tilde{u}(\alpha)$,*

$$\lambda \le \frac{1}{\tau+u}\left(1-\frac{r_2}{(\tilde{u}(0)/u-1)r_0+r_2}\right), \tag{14}$$

and $0<\alpha<u^{-1}-\lambda$. For $x_1=0$ and $x_0\ge 1$, there exists a threshold $x_2^*(x_0)$ such that if $x_2 < x_2^*(x_0)$, then $a^*(\boldsymbol{x}) = Tr$, i.e., a class 0 customer should be triaged; otherwise, $a^*(\boldsymbol{x}) = S0$, i.e., a class 0 customer should be served without triage. Furthermore, $x_2^*(x_0)$ is a non-decreasing function of x_0.

To prove Theorem 3, we first prove the following lemma:

LEMMA 1. *For any policy π_1 that triages a class 0 customer in initial state $\boldsymbol{x} \in \mathcal{S}$ when $u > \tau_0$, there exists a policy π_2 that directly serves that class 0 customer, and for which $V^{\pi_1}(\boldsymbol{x}, T) \geq V^{\pi_2}(\boldsymbol{x}, T)$ for any $T \geq 1$.*

Proof: For any initial state $\boldsymbol{x} \in \mathcal{S}$ with $x_0 \geq 1$, consider policy π_1 that triages a class 0 customer at $t = 0$. We also consider policy π_2 based on π_1 but serves a class 0 customer directly at $t = 0$. There are two possible scenarios: (i) If the service of the class 0 customer under π_2 is not completed at $t = 1$, then in the corresponding sample path, the triage in π_1 does not finish at $t = 1$ because $u > \tau_0$. Hence, the two sample paths couple at $t = 1$ and letting π_2 follow π_1 after $t = 1$, we obtain $V^{\pi_1}(\boldsymbol{x}, T) - V^{\pi_2}(\boldsymbol{x}, T) = 0,\ \forall T \geq 1$. (ii) If the service of the class 0 customer under π_2 finishes at $t = 1$, let π_2 follow π_1 starting from $t = 1$ except when π_1 works (triages or serves directly) on the customer being triaged at $t = 0$, then let policy π_2 stay idle. Hence, $V^{\pi_1}(\boldsymbol{x}, T) - V^{\pi_2}(\boldsymbol{x}, T) \geq 0,\ \forall T \geq 1$, which concludes the proof. □

Based on Lemma 1, we exclude policies that take the action of triage when $u > \tau_0$.

Proof of Theorem 3 (i)&(ii). We first show that (i)&(ii) are optimal for the T-period problem in (11) using stochastic coupling and induction on T. For $T = 1$, the results are true since $V^{\pi}(\boldsymbol{x}, 1) = E\left[\gamma \sum_{i=0}^{2} r_i \boldsymbol{x}\right]$ does not depend on π. Now assume the results hold for some $T \geq 1$. Using stochastic coupling we will show that it also holds for $T + 1$. We first show that $S1$ is better than every other possible action one by one.

(i)-1: Serving a class 1 is better than serving a class 2. Define policy π_1 that serves a class 2 customer (assume there is one) at $t = 0$ while there is a class 1 customer and then follows the optimal policy at $t = 1$. Then, π_1 must serve a class 1 at $t = 1$ by the induction assumption. Consider now the policy π_2 that switches the order of the first two actions under π_1 and then follows π_1 starting at $t = 2$. The difference between the expected cost for the two policies is

$$V^{\pi_1}(\boldsymbol{x}, T+1) - V^{\pi_2}(\boldsymbol{x}, T+1) = \tau_2^{-1}\gamma^2 r_1 + (1 - \tau_2^{-1})\gamma^2(r_1 + r_2) - \left[\tau_1^{-1}\gamma^2 r_2 + (1 - \tau_1^{-1})\gamma^2(r_1 + r_2)\right]$$
$$= \gamma^2(r_1/\tau_1 - r_2/\tau_2) \geq 0.$$

Hence, serving a class 1 customer is better than serving a class 2 customer.

(i)-2: Serving a class 1 is better than serving a class 0. The proof is similar to Case (i)-1 thus omitted.

(i)-3: Serving a class 1 is better than triaging a class 0. Define policy π_1 that triages a class 0 customer (assume there is one) at $t = 0$ while there is a class 1 customer and then follows the optimal policy at $t = 1$. Then, π_1 must serve a class 1 at $t = 1$ by the induction assumption. Consider now the policy π_2 that switches the order of the first two actions under π_1 and then follows π_1 starting at $t = 2$. The difference between the expected cost for the two policies is

$$V^{\pi_1}(\boldsymbol{x}, T+1) - V^{\pi_2}(\boldsymbol{x}, T+1) = (r_0 + r_1)\gamma^2 - \left[\tau_1^{-1}\gamma^2 r_0 + (1 - \tau_1^{-1})\gamma^2(r_0 + r_1)\right] = \gamma^2 r_1 \tau_1^{-1} > 0.$$

Hence, serving a class 1 is better than triaging a class 0 customer.

We have proved that $a^*(\boldsymbol{x}) = S1$ if $x_1 \geq 1$, hence we only need to consider the case when $x_1 = 0, x_0 > 0$, and $x_2 > 0$ for Theorem 3 (ii). Define policy π_1 that serves a class 2 customer at $t = 0$ and then follows the optimal policy at $t = 1$. Then, by the induction assumption, π_1 must work on a class 0, either by serving directly ($S0$) or performing triage (*Tr*) at $t = 1$. Consider policy π_2 that directly serves a class 0 at $t = 0$, serves a class 2 at $t = 1$, and then goes on to follow the optimal policy starting at $t = 2$.

If π_1 takes action $S0$ at $t = 1$, then the difference between the expected cost for the two policies is $V^{\pi_1}(\boldsymbol{x}, T+1) - V^{\pi_2}(\boldsymbol{x}, T+1) = \gamma^2(r_0/\tau_0 - r_2/\tau_2) \geq 0$. If π_1 takes action *Tr* at $t = 1$, then by Lemma 1 we must have $u \leq \tau_0$. Next, we write down the costs under policies π_1 and π_2 as follows.

$$\begin{aligned}
&V^{\pi_1}(\boldsymbol{x}, T+1) \\
=\,& \gamma \boldsymbol{x}\boldsymbol{r} + \gamma^2 \boldsymbol{x}\boldsymbol{r} + \gamma^2 \lambda r_0 - \gamma^2 r_2 \tau_2^{-1} + \gamma^2 \tau_2^{-1} \big[u^{-1} \left[q_1 V(\boldsymbol{x} - \boldsymbol{e_1} + \boldsymbol{e_2} - \boldsymbol{e_3}, T-1) + q_2 V(\boldsymbol{x} - \boldsymbol{e_1}, T-1) \right] \\
&+ \lambda V(\boldsymbol{x} + \boldsymbol{e_1} - \boldsymbol{e_3}, T-1) + (1 - \lambda - u^{-1}) V(\boldsymbol{x} - \boldsymbol{e_3}, T-1) \big] + \gamma^2 \lambda \big[u^{-1} \big[q_1 V(\boldsymbol{x} + \boldsymbol{e_2}, T-1) + q_2 V(\boldsymbol{x} + \boldsymbol{e_3}, T-1) \big] \\
&+ \lambda V(\boldsymbol{x} + 2\boldsymbol{e_1}, T-1) + (1 - \lambda - u^{-1}) V(\boldsymbol{x} + \boldsymbol{e_1}, T-1) \big] + \gamma^2 (1 - \lambda - \tau_2^{-1}) \big[u^{-1} \big[q_1 V(\boldsymbol{x} - \boldsymbol{e_1} + \boldsymbol{e_2}, T-1) \\
&+ q_2 V(\boldsymbol{x} - \boldsymbol{e_1} + \boldsymbol{e_3}, T-1) \big] + \lambda V(\boldsymbol{x} + \boldsymbol{e_1}, T-1) + (1 - \lambda - u^{-1}) V(\boldsymbol{x}, T-1) \big],
\end{aligned} \tag{15}$$

$$\begin{aligned}
&V^{\pi_2}(\boldsymbol{x}, T+1) \\
=\,& \gamma \boldsymbol{x}\boldsymbol{r} + \gamma^2 \boldsymbol{x}\boldsymbol{r} + \gamma^2 \lambda r_0 - \gamma^2 r_0 \tau_0^{-1} \\
&+ \gamma^2 \tau_2^{-1} \left[\tau_0^{-1} V(\boldsymbol{x} - \boldsymbol{e_1} - \boldsymbol{e_3}, T-1) + \lambda V(\boldsymbol{x} + \boldsymbol{e_1} - \boldsymbol{e_3}, T-1) + (1 - \lambda - \tau_0^{-1}) V(\boldsymbol{x} - \boldsymbol{e_3}, T-1) \right] \\
&+ \gamma^2 \lambda \left[\tau_0^{-1} V(\boldsymbol{x}, T-1) + \lambda V(\boldsymbol{x} + 2\boldsymbol{e_1}, T-1) + (1 - \lambda - \tau_0^{-1}) V(\boldsymbol{x} + \boldsymbol{e_1}, T-1) \right] \\
&+ \gamma^2 (1 - \lambda - \tau_2^{-1}) \left[\tau_0^{-1} V(\boldsymbol{x} - \boldsymbol{e_1}, T-1) + \lambda V(\boldsymbol{x} + \boldsymbol{e_1}, T-1) + (1 - \lambda - \tau_0^{-1}) V(\boldsymbol{x}, T-1) \right],
\end{aligned} \tag{16}$$

where $V(\boldsymbol{x}, t)$ is the total discounted cost in the next t periods starting from state $\boldsymbol{x}$ under the optimal policy. It is easy to use a sample-path argument to show that $V(\boldsymbol{x} + \boldsymbol{e_i}, t) \geq V(\boldsymbol{x}, t), \forall \boldsymbol{x} \in \mathcal{S}, i = 1, 2, 3$, and $t \geq 0$. Hence, from (15) and (16) and the fact that $u \leq \tau_0$, we get $V^{\pi_1}(\boldsymbol{x}, T+1) - V^{\pi_2}(\boldsymbol{x}, T+1) > \gamma^2(r_0/\tau_0 - r_2/\tau_2) \geq 0$. Since the results hold for all $T < \infty$, they also hold for the infinite-horizon problem with the total discounted cost, which concludes our proof. □

We need the following lemma to prove Theorem 3 (iii).

LEMMA 2. *Assume $r_1/\tau_1 \geq r_0/\tau_0 \geq r_2/\tau_2$. For $\boldsymbol{x} = (0, 0, x_2)$, we have (i) $v(\boldsymbol{x} + \boldsymbol{e_2}) - v(\boldsymbol{x}) \geq (r_1 + x_2 r_2)\tau_1/(1 + \alpha\tau_1)$; and (ii) $v(\boldsymbol{x} + \boldsymbol{e_3}) - v(\boldsymbol{x}) \geq (r_2 + x_2 r_2)\tau_2/(1 + \alpha\tau_2)$, where $v(\boldsymbol{x})$ is the minimum total discounted cost starting in state $\boldsymbol{x}$.*

Proof: By Theorem 3 (i), the optimal action at state $\boldsymbol{x} + \boldsymbol{e_2}$ is to *serve class 1*. Hence,

$$v(\boldsymbol{x} + \boldsymbol{e_2}) - v(\boldsymbol{x}) = \lambda v(\boldsymbol{x} + \boldsymbol{e_1} + \boldsymbol{e_2}) + \tau_1^{-1} v(\boldsymbol{x}) + (u^{-1} + \tau_0^{-1} + \tau_2^{-1}) v(\boldsymbol{x} + \boldsymbol{e_2}) + r_1 + x_2 r_2 - v(\boldsymbol{x})$$

$$> (1-\alpha-\tau_1^{-1})(v(\boldsymbol{x}+\boldsymbol{e_2})-v(\boldsymbol{x}))+r_1+x_2r_2,$$

which yields $v(\boldsymbol{x}+\boldsymbol{e_2})-v(\boldsymbol{x}) > (r_1+x_2r_2)\tau_1/(1+\alpha\tau_1)$. Similarly, we have

$$v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}) = \lambda v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})+\tau_2^{-1}v(\boldsymbol{x})+(u^{-1}+\tau_0^{-1}+\tau_1^{-1})v(\boldsymbol{x}+\boldsymbol{e_3})+r_2+x_2r_2-v(\boldsymbol{x})$$
$$> (1-\alpha-\tau_2^{-1})(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))+r_2+x_2r_2,$$

which yields $v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}) > (r_2+x_2r_2)\tau_2/(1+\alpha\tau_2)$. □

Proof of Theorem 3 (iii). We prove the result by a sample-path argument for the uniformized discrete-time process. By Lemma 1, we only need to consider cases with $u \le \tau_0$. For initial system state $\boldsymbol{x} = (1,0,x_2),\ x_2 \ge 0$, consider policy π_1 that triages the single class 0 customer at $t=0$ and follows the optimal policy starting from $t=1$, and policy π_2 that serves the single class 0 customer at $t=0$ and follows the optimal policy starting from $t=1$. There are three possible scenarios:

(i) With probability τ_0^{-1}, the service of the class 0 customer under π_2 finishes by $t=1$, then in the corresponding sample path, the triage under π_1 must be also complete by $t=1$ (because $u \le \tau_0$). Hence,

$$V^{\pi_1}(\boldsymbol{x})-V^{\pi_2}(\boldsymbol{x}) = q_1(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})-v(\boldsymbol{x}-\boldsymbol{e_1}))+q_2(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))$$
$$\ge q_1(r_1+x_2r_2)\tau_1/(1+\alpha\tau_1)+q_2(r_2+x_2r_2)\tau_2/(1+\alpha\tau_2),$$

where the inequality follows from Lemma 2.

(ii) With probability $u^{-1}-\tau_0^{-1}$, the service of the class 0 customer under π_2 is not complete by $t=1$ and hence the state is still $\boldsymbol{x}$, however, the triage under π_1 is complete by $t=1$. Hence, we have

$$V^{\pi_1}(\boldsymbol{x})-V^{\pi_2}(\boldsymbol{x}) = V^{\pi_1}(\boldsymbol{x})-(r_0+x_2r_2+\gamma v(\boldsymbol{x})) \ge V^{\pi_1}(\boldsymbol{x})-v(\boldsymbol{x})-(r_0+x_2r_2) > -(r_0+x_2r_2)u,$$

where the first inequality holds because $\gamma \le 1$ and the last one holds because $V^{\pi_1}(\boldsymbol{x}) \ge v(\boldsymbol{x})$ and $u > 1$.

(iii) With probability $1-u^{-1}$, the triage under π_1 is not complete by $t=1$, nor is the service of the class 0 customer under π_2 (because $u \le \tau_0$). The event that happened in $t=1$ could be a dummy transition due to the uniformization or a new arrival, and thus, in either case, the two sample paths under π_1 and π_2 have the same states starting from $t=1$ and accumulated the same cost in the previous period. Hence, $V^{\pi_1}(\boldsymbol{x})-V^{\pi_2}(\boldsymbol{x}) = 0$.

Taking the expectation over these three possible outcomes at $t=0$, we have

$$V^{\pi_1}(\boldsymbol{x})-V^{\pi_2}(\boldsymbol{x}) > \tau_0^{-1}[q_1(r_1+x_2r_2)\tau_1/(1+\alpha\tau_1)+q_2(r_2+x_2r_2)\tau_2/(1+\alpha\tau_2)]-(u^{-1}-\tau_0^{-1})(r_0+x_2r_2)u$$
$$= \tau_0^{-1}\big[q_1r_1\tau_1/(1+\alpha\tau_1)+q_2r_2\tau_2/(1+\alpha\tau_2)+r_0u-r_0\tau_0\big]+\tau_0^{-1}x_2r_2\big[q_1\tau_1/(1+\alpha\tau_1)+q_2\tau_2/(1+\alpha\tau_2)+u-\tau_0\big] \ge 0,$$

where the last inequality holds by the conditions given in the theorem. □

To prove Theorem 3 (iv), we need Lemma 3, which is proved in the E-companion, but first let $\mathbb{F}$ be the set of functions defined on $\mathcal{S}$ such that if $v \in \mathbb{F}$, then following hold:

f1) $q_1u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2})-v(\boldsymbol{x}))-\tau_0^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))]\le r_0\tau_0^{-1},\ x_0\ge 1,\ x_1=0,\ x_2\ge 0.$

f2) $G(\boldsymbol{x})\ge 0,\ x_0\ge 1,\ x_1=0,\ x_2\ge 0$, where

$$G(\boldsymbol{x})\equiv u^{-1}[q_1v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})+q_2v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+\tau_0^{-1}v(\boldsymbol{x})-u^{-1}v(\boldsymbol{x})-\tau_0^{-1}v(\boldsymbol{x}-\boldsymbol{e_1}). \tag{17}$$

Lemma 3. *Suppose that* $\frac{r_1}{\tau_1}\ge\frac{r_0}{\tau_0}\ge\frac{r_2}{\tau_2}$, $q_1r_1\tau_1/(1+\alpha\tau_1)+q_2r_2\tau_2/(1+\alpha\tau_2)+r_0u\ge r_0\tau_0$, $q_1\tau_1/(1+\alpha\tau_1)+q_2\tau_2/(1+\alpha\tau_2)+u\ge\tau_0$, *and* $u\ge\tilde{u}(\alpha)$. *(i) If* $v\in\mathbb{F}$, *then* $Lv\in\mathbb{F}$. *(ii) There exists an optimal stationary policy with a value function that possesses the properties of functions from set* $\mathbb{F}$.

Proof of Theorem 3 (iii). Lemma 3 proved that there exist an optimal stationary policy and the corresponding (optimal) value function that satisfies property f2) when $u\ge\tilde{u}(\alpha)$. Property f2) implies that it is better to directly serve a class 0 customer than to triage it at any system state when $u\ge\tilde{u}(\alpha)$, hence the result. □

To prove Theorem 3 (v), we need Lemma 4, which is proved in the E-companion, but first let $\mathbb{H}$ be the set of functions defined on $\mathcal{S}$ such that if $v(\cdot)\in\mathbb{H}$, then following hold when $\tau_0=\tau_1=\tau_2=\tau$:

h1) $G(\boldsymbol{x})\le G(\boldsymbol{x}+\boldsymbol{e_3}),\ x_0\ge 1,\ x_1=0,\ x_2\ge 0.$

h2) $v(\boldsymbol{x}+\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_2})\le v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_2}+\boldsymbol{e_3}),\ x_0\ge 0,\ x_1=0,\ x_2\ge 0.$

h3) $v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})\le v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}),\ x_0\ge 0,\ x_1=0,\ x_2\ge 0.$

h4) $v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})\ge v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}),\ x_0\ge 1,\ x_1=0,\ x_2\ge 0.$

h5) $G(\boldsymbol{x})\ge G(\boldsymbol{x}+\boldsymbol{e_1}),\ x_0\ge 1,\ x_1=0,\ x_2\ge 0.$

h6) $v(\boldsymbol{x}+\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_2})\ge v(\boldsymbol{x}+2\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}),\ x_0\ge 0,\ x_1=0,\ x_2\ge 0.$

h7) $q_1[v(\boldsymbol{x}+\boldsymbol{e_2})-v(\boldsymbol{x}+\boldsymbol{e_1})]\ge\tilde{u}(\alpha)r_0,\ x_0\ge 0,\ x_1=0,\ x_2\ge 0.$

h8) $G(\boldsymbol{x})-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]\le\tilde{u}(\alpha)u^{-1}r_0,\ x_0\ge 1,\ x_1=0,\ x_2\ge 1.$

h9) $[v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3})]-[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]\le(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2),\ \forall\boldsymbol{x}\in\mathcal{S}.$

h10) $G(x_0,0,0)\le r_0,\ x_0\ge 1.$

h11) For any $\boldsymbol{x}\in\mathcal{S}$, $v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})\le r_2\alpha^{-1}\left(1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}\right)$, where

$$\beta_1\equiv\frac{\lambda(1+u\tau^{-1})+\tau^{-1}+\alpha-\sqrt{[\lambda(1+u\tau^{-1})+\tau^{-1}+\alpha]^2-4\lambda(1+u\tau^{-1})\tau^{-1}}}{2\lambda(1+u\tau^{-1})}, \tag{18}$$

$$\beta_2\equiv(1+u\tau^{-1})\beta_1-u\tau^{-1}. \tag{19}$$

Lemma 4. *Suppose that* (14) *holds,* $r_1\ge r_0\ge r_2$, $\tau_0=\tau_1=\tau_2=\tau$, $\alpha\tau^2/(1+\alpha\tau)<u<\tilde{u}(\alpha)$, *and* $0<\alpha<u^{-1}-\lambda$. *(i) If* $v\in\mathbb{H}$, *then* $Lv\in\mathbb{H}$. *(ii) There exists an optimal stationary policy with a value function that possesses the properties of functions from set* $\mathbb{H}$.

Proof of Theorem 3 (v). Lemma 4 proved that there exist an optimal stationary policy and the corresponding (optimal) value function that satisfies properties properties h1) and h5) when $r_1\ge r_0\ge r_2,\ \tau_0=\tau_1=\tau_2=\tau,\ \alpha\tau^2/(1+\alpha\tau)<u<\tilde{u}(\alpha)$, and $0<\alpha<u^{-1}-\lambda$. Properties h1) and h5) imply that the optimal policy on

whether to triage or not is determined by a threshold $x_2^*(x_0)$ for any given x_0, which is a non-decreasing function of x_0. To be more specific, if it is optimal to skip triage in $(x_0, 0, x_2)$ for $x_0 \geq 1$ and $x_2 \geq 0$, then it is optimal to do so in $(x_0', 0, x_2')$ for $1 \leq x_0' \leq x_0$ and $x_2' \geq x_2$. This completes the proof of Theorem 3 (v). □

Appendix B. Proofs of Proposition 1 and Theorem 1

We first show that the three SEN conditions given in Section 7.2 of Sennott (1999) hold. Let z be some state in $\mathcal{S}$, $V_\gamma^\pi(z)$ be the total discounted cost under policy π starting from z, and $V_\gamma(z)$ be the optimal total discounted cost. The SEN conditions are as follows:

SEN1 The quantity $(1-\gamma)V_\gamma(z)$ is bounded for $\gamma \in (0,1)$.

SEN2 There exists a nonnegative (finite) function $M(\cdot)$ such that $h_\gamma(\boldsymbol{x}) \equiv V_\gamma(\boldsymbol{x}) - V_\gamma(z) \leq M(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{S}$ and $\gamma \in (0,1)$.

SEN3 There exists a nonnegative (finite) constant K such that $-K \leq h_\gamma(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{S}$ and $\gamma \in (0,1)$.

Note that after uniformization, the discounting factor for the equivalent discrete-time MDP is $\gamma = 1-\alpha$; see Figure 11.5.3 in Puterman (2005). Hence, letting $\alpha \to 0$ is equivalent to letting $\gamma \to 1$.

LEMMA 5. *Assume that $\lambda\tau_0 < 1$. The three SEN conditions are satisfied for the infinite horizon total discounted cost problem in* (10).

Proof: We first verify SEN 1. Let $z = \mathbf{0}$ be the initial system state and π be the policy that serves all class 0 customers directly without triage. Hence, this is an $M/M/1$ queue with arrival rate λ and service rate τ_0^{-1} starting at the empty and idle state. Thus, we have

$$V_\gamma^\pi(\mathbf{0}) = E\Big[\int_0^\infty e^{-(1-\gamma)t} r_0 Q(t)dt\Big|Q(0)=0\Big] = r_0\int_0^\infty e^{-(1-\gamma)t}E[Q(t)|Q(0)=0]dt \leq \frac{r_0\lambda\tau_0}{(1-\gamma)(1-\lambda\tau_0)},$$

where $Q(t)$ is the number of customers in the $M/M/1$ queue at time t. The above inequality follows from Corollary 3.1.1 of Abate and Whitt (1987) that $E[Q(t)|Q(0)=0]$ is non-decreasing in t and bounded by $\lim_{t\to\infty} E[Q(t)|Q(0)=0] = \lambda\tau_0/(1-\lambda\tau_0)$ where $\lambda\tau_0 < 1$. Hence, $(1-\gamma)V_\gamma(\mathbf{0}) \leq (1-\gamma)V_\gamma^\pi(\mathbf{0}) \leq \frac{r_0\lambda\tau_0}{1-\lambda\tau_0} < \infty, \forall\gamma \in (0,1)$, i.e., SEN1 holds. To prove SEN 2, we need to find an upper bound on $h_\gamma(\boldsymbol{x}) = V_\gamma(\boldsymbol{x}) - V_\gamma(\mathbf{0})$ for all $\boldsymbol{x} = (x_0, x_1, x_2) \in \mathcal{S}$. Let $M(\mathbf{0}) = 0$ and for $\boldsymbol{x} \in \mathcal{S}\backslash\{\mathbf{0}\}$ define π' as a policy that serves customers in FCFS manner with no triage until the state reaches $\mathbf{0}$, after which it follows the optimal policy. Let $T_0(\boldsymbol{x})$ be the first passage time to state $\mathbf{0}$ starting from state $\boldsymbol{x}$ and $N_0(t)$ be the total number of customers served during $[0, T_0(\boldsymbol{x})]$ under π'. Then, $h_\gamma(\boldsymbol{x}) \leq V_\gamma^{\pi'}(\boldsymbol{x}) - V_\gamma(\mathbf{0}) \leq \max\{r_0, r_1, r_2\}E[T_0(\boldsymbol{x})N_0(\boldsymbol{x})]$. Since $\lambda\tau_0 < 1$, it can be shown that $M(\boldsymbol{x}) \equiv E[T_0(\boldsymbol{x})N_0(\boldsymbol{x})] \geq 0$ is a finite function of $\boldsymbol{x}$, which completes the verification of SEN2. We finally prove SEN3 by a simple sample-path argument. Consider two systems. System 1 and System 2 are identical except that System 1 starts in state $\boldsymbol{x}$ and uses the optimal policy and System 2 starts in state $\mathbf{0}$ and uses policy $\tilde{\pi}$, which takes whatever action System 1 takes if possible; otherwise, it idles. Then, we have $h_\gamma(\boldsymbol{x}) \geq V_\gamma(\boldsymbol{x}) - V_\gamma^{\tilde{\pi}}(\mathbf{0}) \geq 0$, which completes the proof.□

Proofs of Proposition 1 and Theorem 1. By Lemma 5, Theorem 7.2.3 (ii) in Sennott (1999) implies that there exists an optimal stationary policy with optimal bias function $h(\cdot)$ and constant average cost g^* satisfying the inequalities in (5). Hence, Proposition 1 holds. Letting $\alpha \to 0$ (equivalently $\gamma \to 1$) in the proof of Theorem 3 (i)(ii)&(iii) proves the results for parts (i), (ii), and partially for (iii). Furthermore, Theorem 7.2.3 (ii) in Sennott (1999) implies that $h(\boldsymbol{x})$ is a limit function of the sequence $h_{\alpha_n}(\boldsymbol{x}) \equiv V_{\alpha_n}(\boldsymbol{x}) - V_{\alpha_n}(\boldsymbol{z})$, i.e., $\lim_{n\to\infty} h_{\alpha_n}(\boldsymbol{x}) = h(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathcal{S}$, where $0 < \alpha_n < u^{-1} - \lambda$, $\boldsymbol{z}$ is some state in $\mathcal{S}$, and $\alpha_n \downarrow 0$ as $n \to \infty$. Hence, $h_{\alpha_n}(\boldsymbol{x})$ inherits all the properties of the optimal value function of the discounted-cost problem, and so does $h(\boldsymbol{x})$. To be more specific, $h(\boldsymbol{x}) \in \mathbb{F}$ if $u \geq \tilde{u}(0)$, and $h(\boldsymbol{x}) \in \mathbb{H}$ if $u < \tilde{u}(0)$ and condition (7) holds, where the right-hand sides of h7), h8), and h11) hold for $\alpha \downarrow 0$, i.e.,

h7) $q_1[v(\boldsymbol{x}+\boldsymbol{e_2}) - v(\boldsymbol{x}+\boldsymbol{e_1})] \geq \tilde{u}(0)r_0,\ x_0 \geq 0,\ x_1 = 0,\ x_2 \geq 0;$

h8) $G(\boldsymbol{x}) - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] \leq \tilde{u}(0)u^{-1}r_0,\ x_0 \geq 1,\ x_1 = 0,\ x_2 \geq 1;$

h11) $v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x}) \leq \frac{r_2}{1-\lambda(\tau+u)}[\tau(x_0+x_1+x_2+1)+ux_0],\ \boldsymbol{x} \in \mathcal{S}.$

Hence, as in the proof of Theorem 3 (v), see last paragraph of Appendix A, we conclude that the optimal stationary policy has the structural properties described in Theorem 3 (iv)&(v). Hence, Theorem 1 holds. □

Appendix C. Proofs of Propositions 2 and 3, Theorem 2, and Corollary 1

Let Π' denote the set of deterministic policies under which all class 0 customers are triaged (not necessarily upon arrival). Our next result describes the best policy in Π', which will be used to prove Proposition 2. To simplify the presentation of the proofs, let $\rho_0 \equiv \lambda(u + q_1\tau_1)$.

LEMMA 6. *Assume that* $\frac{r_1}{\tau_1} \geq \frac{r_2}{\tau_2}$ *and* $\rho < 1$. *The following policy minimizes the expected long-run average cost within* Π'*: (i) Class 1 customers have the highest priority. (ii) When there are no class 1 customers, triaging a class 0 customer is preferable over serving a class 2 if* $r_0\tau_2 > r_2(u + q_1\tau_1 + q_2\tau_2)$*, and serving a class 2 customer is preferable over triaging a class 0 otherwise.*

Proof: When every class 0 customer is required to be triaged and triage/services are preemptive, our problem becomes a special case of the scheduling problem described in Corollary 3 in Lai and Ying (1988), with transition matrix P, arrival rate and service time vectors defined as follows:

$$P = \begin{pmatrix} 0 & q_1 & q_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \boldsymbol{\lambda} = (\lambda, 0, 0),\ \text{and}\ \ \boldsymbol{\mu} = (u, \tau_1, \tau_2).$$

It is easy to verify that the eigenvalues of P are all 0, and $\boldsymbol{\lambda}(I-P)^{-1}\boldsymbol{\mu}' = \lambda(u + q_1\tau_1 + q_2\tau_2) = \rho < 1$. Let ϕ_i be Klimov's index for class $i \in \{0,1,2\}$. By the procedures in Klimov (1974), we get $\phi_0 = r_0/(u + q_1\tau_1 + q_2\tau_2)$, $\phi_1 = r_1/\tau_1$, and $\phi_2 = r_2/\tau_2$. Corollary 3 states that it is optimal to serve customers according to a non-increasing order of Klimov's priority indices. By the assumptions, we have $\phi_1 > \phi_0$ and $\phi_1 > \phi_2$, which proves (i). Conditions in part (ii) are equivalent to the order of ϕ_0 and ϕ_2. □

Proof of Proposition 2. From Lemma 6, we know that the optimal policy within Π' is *TP1* when $r_0\tau_2 > r_2(u + q_1\tau_1 + q_2\tau_2)$. If $r_0\tau_2 \leq r_2(u + q_1\tau_1 + q_2\tau_2)$, then a policy that triages every class 0 customer and serves the customer immediately after triage is optimal. We call it policy *TS*. The stochastic system under policy *TS* is an M/G/1 queue with a phase-type distributed service time whose first and second moments are $u + q_1\tau_1 + q_2\tau_2$ and $2[u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u)]$, respectively. Hence, the long-run average number of customers in this queue, denoted by Q_{TS}, is given by

$$Q_{TS} = \lambda^2(u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u))/(1 - \rho), \tag{20}$$

and the long-run average cost is $c_{TS} = Q_{TS}r_0 + \lambda(ur_0 + q_1r_1\tau_1 + q_2r_2\tau_2)$. Hence,

$$c_{TS} - c_{NT} = \lambda^2 r_0 \left(\frac{u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u)}{1 - \rho} - \frac{\tau_0^2}{1 - \lambda\tau_0} \right) + \lambda(r_0u + q_1r_1\tau_1 + q_2r_2\tau_2 - r_0\tau_0) \geq 0, \tag{21}$$

where the inequality holds because of $u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u) \geq \tau_0^2$, $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$, and $r_0u + q_1r_1\tau_1 + q_2r_2\tau_2 \geq r_0\tau_0$. Hence, within the set of all deterministic state-independent policies, it is sufficient to consider only *TP1* and *NT*. □

Proof of Proposition 3. The stochastic system under *TP1* contains two queues in steady state: we call the queue where a class 0 customer waits for triage (and service if it is classified as class 1) *Queue 0*, and call the queue where a class 2 customer waits for service after it is triaged *Queue 2*. Let Q_0 denote the long-run average number of customers waiting in Queue 0 and L_2 denote the long-run average number of class 2 customers in the system (including those in Queue 2 and in service). Since the server gives preemptive priority to triaging class 0 customers and serving class 1 customers over serving class 2 customers, to obtain Q_0, we can use the steady-state number in the queue formula for an M/G/1 queue with a phase-type service time distribution having a mean of $u + q_1\tau_1$ and a second moment of $2(u^2 + q_1u\tau_1 + q_1\tau_1^2)$, see, e.g., Theorem 7.12 in Kulkarni (2010). Hence, $Q_0 = \lambda^2(u^2 + q_1u\tau_1 + q_1\tau_1^2)/(1 - \rho_0)$, and the expected remaining service time of the customer in service observed at a random time in steady state, denoted by $E(R_0)$, can be obtained as $E(R_0) = \lambda(u^2 + q_1u\tau_1 + q_1\tau_1^2)$, see, e.g., Problem 5.7 in Gross et al. (2008). Note that both *TP1* and *TS* are work-conserving policies, i.e., no service needs are created or destroyed within the system under each of the two policies. Then, by the invariance of expected long-run average workload of work-conserving policies, see, e.g., Theorem 1 in Chapter 10 of Wolff (1989), we have

$$E(R_0) + Q_0(u + q_1\tau_1 + q_2\tau_2) + L_2\tau_2 = E(R_{TS}) + Q_{TS}(u + q_1\tau_1 + q_2\tau_2), \tag{22}$$

where Q_{TS} is given by (20), and the expected remaining service time of the customer in service, denoted by $E(R_{TS})$, is given by $E(R_{TS}) = \lambda(u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u))$. Hence, we have

$$L_2 = \lambda q_2 \left[u + \tau_2 + \frac{\rho[u + \tau_2 - \lambda u\tau_2 - \lambda q_1\tau_1(\tau_2 - \tau_1)]}{(1 - \rho)(1 - \rho_0)} \right]. \tag{23}$$

With the expressions Q_0 and L_2, we have $c_{TPI} = Q_0 r_0 + \lambda(ur_0 + q_1\tau_1 r_1) + L_2 r_2$. $\square$

Proof of Theorem 2. Let $f_1(\lambda) \equiv (c_{NT} - c_{TPI})(1 - \lambda\tau_0)(1 - \rho)(1 - \rho_0)/\lambda$. We prove part (i) by showing that $f_1(\lambda) = 0$ has at most two solutions in $(0,\ (u + q_1\tau_1 + q_2\tau_2)^{-1})$, $\lim_{\lambda\to 0} f_1(\lambda) < 0$, and $\lim_{\lambda\to(u+q_1\tau_1+q_2\tau_2)^{-1}} f_1(\lambda) \le 0$. We first plug into $f_1(\lambda)$ the expressions of c_{NT} and c_{TPI} in (8) and (9), respectively; then we rewrite $f_1(\lambda)$ as $f_1(\lambda) = A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$, where

$$A_3 = u\tau_0(u + q_1\tau_1 + q_2\tau_2)[q_2 r_2 u + q_1\tau_1(q_2 r_2 + r_1)] > 0,$$

$$A_0 = -[r_0 u + q_1 r_1 \tau_1 + q_2 r_2 \tau_2 - r_0\tau_0] - q_2 r_2 u < 0.$$

Hence, $f_1(\lambda)$ is a cubic function of λ, and thus is continuous over $(0,\ (u + q_1\tau_1 + q_2\tau_2)^{-1})$. Furthermore, the derivative of $f_1(\lambda)$ is $f_1'(\lambda) = 3A_3\lambda^2 + 2A_2\lambda + A_1$, and hence, $f_1(\lambda)$ can have at most two stationary points. Since $\lim_{\lambda\to 0} f_1(\lambda) = A_0 < 0$, and

$$\lim_{\lambda\to(u+q_1\tau_1+q_2\tau_2)^{-1}} f_1(\lambda) = -\frac{q_2 r_2(u + q_1\tau_1 + q_2\tau_2 - \tau_0)[u^2 + q_1\tau_1(u + \tau_1) + q_2\tau_2(u + \tau_2)]}{(u + q_1\tau_1 + q_2\tau_2)^2} \le 0,$$

we conclude that $f_1(\lambda) = 0$ can have at most two solutions in $(0, (u + q_1\tau_1 + q_2\tau_2)^{-1})$, which completes the proof of part (i).

To prove part (ii), we rewrite $f_1(\lambda)$ as a function of p_1, denote it by $f_2(p_1)$, and obtain $f_2(p_1) \equiv B_3 p_1^3 + B_2 p_1^2 + B_1 p_1 + B_0 = f_1(\lambda)$, where

$$B_3 = \lambda^2\tau_1(\tau_1 - \tau_2)(\theta_1 + \theta_2 - 1)^2[(h_1 - h_2)(\tau_0 - \theta_2 u(1 - \lambda\tau_0)) + (\theta_1 + \theta_2 - 1)h_1 u(1 - \lambda\tau_0)].$$

The derivative of $f_2(p_1)$ is $f_2'(p_1) = 3B_3 p_1^2 + 2B_2 p_1 + B_1$. Hence, $f_2(p_1)$ can have at most two stationary points. We only need to show $\lim_{p_1\to 1} f_2(p_1) < 0$ and $\lim_{p_1\to 0} f_2(p_1) < 0$, which are equivalent to $c_{NT} < c_{TPI}$ when $p_1 \to 1$ and $p_1 \to 0$, respectively.

$$\begin{aligned}\lim_{p_1\to 1}(c_{TPI} - c_{NT}) &= \lim_{p_1\to 1}(c_{TPI} - c_{TS} + c_{TS} - c_{NT})\\ &\ge \lim_{p_1\to 1}(c_{TPI} - c_{TS}) = \lim_{p_1\to 1}[(u + q_1\tau_1 + q_2\tau_2 - \tau_2)\tau_2^{-1}(Q_{TS} - Q_0) + \lambda q_2 u]h_1 > 0,\end{aligned}$$

where the first inequality holds because of (21), the last inequality holds because $u + q_1\tau_1 + q_2\tau_2 \ge \tau_0$, $\tau_2 = \tau_0$, and

$$Q_{TS} - Q_0 = \frac{\lambda^2[u^2 + q_1\tau_1(\tau_1 + u) + q_2\tau_2(\tau_2 + u)]}{1 - \lambda(u + q_1\tau_1 + q_2\tau_2)} - \frac{\lambda^2[u^2 + q_1\tau_1(\tau_1 + u)]}{1 - \lambda(u + q_1\tau_1)} > 0.$$

The proof for $p_1 \to 0$ is similar thus omitted, which completes the proof of part (ii).

To prove part (iii), we write $f_1(\lambda)$ as $f_1(\lambda) = q_1 r_1(1 - \rho)[\lambda q_2\tau_0\tau_1 - (1 - \lambda\tau_1 + \lambda^2\tau_0\tau_1)u + (\tau_0 - \tau_1)] - q_2 r_2 f_3(u)$, where

$$f_3(u) \equiv (1 - \lambda\tau_0)[\tau_2 + (1 - \lambda\tau_2)u - \lambda q_1\tau_1(\tau_2 - \tau_1)] - (1 - \rho)(1 - \rho_0)[\tau_0 - (1 - \lambda\tau_0)u]$$

$$\geq (1-\rho)\{\tau_2 + (1-\lambda\tau_2)u - \lambda q_1\tau_1(\tau_2-\tau_1) - (1-\rho_0)[\tau_0 - (1-\lambda\tau_0)u]\}$$
$$= (1-\rho)\left\{(1-\lambda\tau_0)u[2-\lambda(u+q_1\tau_1)] + \lambda(u\tau_0 + q_1\tau_1^2)\right\},$$

where the first inequality holds because $u + q_1\tau_2 + q_2\tau_2 \geq \tau_0$ and $\rho < 1$, and the last equality holds because $\tau_0 = \tau_2$. Hence, $f_3(u) > 0$ for all $u > 0$ such that $\rho < 1$. This means that $c_{NT} \leq c_{TP1}$ if and only if

$$\frac{r_2/\tau_2}{r_1/\tau_1} \geq q_1\tau_1(q_2\tau_2)^{-1}(1-\rho)[\lambda q_2\tau_0\tau_1 - (1-\lambda\tau_1+\lambda^2\tau_0\tau_1)u + (\tau_0-\tau_1)]/f_3(u),$$

which is equivalent to $\frac{r_2/\tau_2}{r_1/\tau_1} \geq \Theta_1$ since $\frac{r_2/\tau_2}{r_1/\tau_1} > 0$. □

Proof of Corollary 1. We start by showing that $\Theta_2 \leq 1$. First, consider the case where $u \geq \tau_0 - q_1\tau_1$. In this case, it is easy to see that $\Theta_2 = 0$. Otherwise, when $u < \tau_0 - q_1\tau_1$, we have

$$\Theta_2 - 1 = \frac{-q_1\tau_1(u + q_1\tau_1 + q_2\tau_2 - \tau_0) - q_2\tau_2 u}{q_2\tau_2(q_1\tau_1 + u)} < 0,$$

where the inequality holds because $u + q_1\tau_1 + q_2\tau_2 \geq \tau_0$ and $u > 0$. Now, part (i) follows from Theorem 1 (iv), whereas parts (ii) and (iii) follow from Theorem 2 (iii). Note that $\Theta_2 \geq \Theta_1$ holds, otherwise, there exists a $\frac{r_2/\tau_2}{r_1/\tau_1}$ such that $\Theta_2 < \frac{r_2/\tau_2}{r_1/\tau_1} < \Theta_1$, i.e., *TP1* is the best static deterministic state-independent policy (see the proof of Theorem 2 (iii) where $c_{NT} > c_{TP1}$), which contradicts with the fact that *NT* is the optimal policy by Theorem 1. □

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Proof of Lemma 3. We first prove part (i), i.e., f1) and f2) are preserved under operator L. For any $\boldsymbol{x} \in \mathcal{S}$ and $x_0 \geq 1, x_1 = 0$, by Theorem 3 (i)&(ii) and f2), we have

$$\begin{aligned}
& q_1 u^{-1}[\tau_1^{-1}(Lv(\boldsymbol{x}+\boldsymbol{e_2}) - Lv(\boldsymbol{x})) - \tau_0^{-1}(Lv(\boldsymbol{x}+\boldsymbol{e_2}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))] \\
&= \lambda q_1 u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}+\boldsymbol{e_1})) - \tau_0^{-1}(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}+\boldsymbol{e_2}))] + q_1 u^{-1}(r_1/\tau_1 - r_0/\tau_0) \\
&\quad + (u^{-1}+\tau_0^{-1}+\tau_2^{-1}) q_1 u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2}) - v(\boldsymbol{x})) - \tau_0^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))] \\
&\leq (1-\alpha-\tau_1^{-1}) r_0 \tau_0^{-1} + q_1 u^{-1}(r_1/\tau_1 - r_0/\tau_0) = r_0\tau_0^{-1} + (1+\alpha\tau_1)(\tilde{u}(\alpha) - u) r_0/(\tau_0\tau_1 u) \leq r_0\tau_0^{-1},
\end{aligned}$$

where the first inequality holds because of f1) and the last inequality holds because of the assumption $u \geq \tilde{u}(\alpha)$, which completes the proof that f1) is preserved under L.

Next, we show that f2) is preserved under operator L. For $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \geq 2$ and $x_2 \geq 0$, by Theorem 3 (i)&(ii) and f2), we have

$$\begin{aligned}
LG(\boldsymbol{x}) &= \lambda G(\boldsymbol{x}+\boldsymbol{e_1}) + (u^{-1}+\tau_1^{-1}+\tau_2^{-1}) G(\boldsymbol{x}) + \tau_0^{-1} G(\boldsymbol{x}-\boldsymbol{e_1}) \\
&\quad + r_0\tau_0^{-1} - q_1 u^{-1}[(\tau_1^{-1}-\tau_0^{-1})[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})] + \tau_0^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})]] \\
&\geq r_0\tau_0^{-1} - q_1 u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})) - \tau_0^{-1}(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}))] \geq 0,
\end{aligned}$$

where the first inequality holds because of f2) and Theorem 3 (iii), and the second inequality holds because of f1), which completes the proof that f2) is preserved under L.

We prove part (ii) by verifying the conditions in Theorem 6.11.3 of Puterman (2005). It is obvious that the state space $\mathcal{S}$ is countable and we have already verified Assumptions 6.10.1 and 6.10.2 of Puterman (2005) in our proof of Proposition 4. Hence, we only need to show that conditions (a), (b), and (c) in Theorem 6.11.3 of Puterman (2005) hold. Condition (a) holds by part (i) of Lemma 3. Next, consider a stationary policy π that never triages class 0 customers and serves customers in the following priority order: class 1, class 0, and class 2. From the optimality equations, we find that $v \in \mathbb{F}$ implies that policy π is an optimal policy. Hence, (b) holds. Finally, condition (c) holds, i.e., $\mathbb{F}$ is closed, because the limit of any convergent sequence of functions that satisfy f1) and f2), will satisfy them as well, which concludes the proof. □

We need the following lemma to prove Lemma 4.

LEMMA EC.1. *Suppose that* (14) *holds,* $r_1 \geq r_0 \geq r_2,\ \tau_0 = \tau_1 = \tau_2 = \tau,\ u < \tilde{u}(\alpha)$, *and* $0 < \alpha < u^{-1} - \lambda$. *Then* β_1 *and* β_2, *as defined in* (18) *and* (19), *satisfy the following conditions:*

$$\frac{r_2}{\alpha}(1-\beta_1) \leq (q_2 u^{-1} - \tau^{-1})(r_0 - r_2)\tau^2, \tag{EC.1}$$

$$0 \le \beta_2 < \beta_1 < 1, \tag{EC.2}$$

$$\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1 = 0, \tag{EC.3}$$

$$\lambda\beta_2(1-\beta_2) - \tau^{-1}(1-\beta_2) + \alpha\beta_2 \le 0, \tag{EC.4}$$

$$\lambda\beta_2(1-\beta_2) - u^{-1}(\beta_1-\beta_2) + \alpha\beta_2 \le 0. \tag{EC.5}$$

Proof: We can rewrite (EC.1) as $\beta_1 \ge 1 - \frac{\alpha\tau^2}{r_2}(q_2u^{-1} - \tau^{-1})(r_0 - r_2)$, whose right-hand-side is denoted by f. By (14) and the assumption that $u < \tilde{u}(\alpha)$, we know $\lambda(1+u/\tau) < \tau^{-1}$. Then, it is easy to show that $\beta_1 = f = 1$ when $\alpha = 0$, and

$$\frac{d\beta_1}{d\alpha} = \frac{1}{2\lambda(1+u\tau^{-1})}\left[1 - \frac{\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha}{\sqrt{(\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha)^2 - 4\lambda(1+u\tau^{-1})\tau^{-1}}}\right] < 0,$$

$$\left.\frac{d\beta_1}{d\alpha}\right|_{\alpha=0} = -\frac{1}{\tau^{-1} - \lambda(1+u\tau^{-1})}, \text{ and } \frac{df}{d\alpha} = -\frac{\tau^2}{r_2}\left(q_2u^{-1} - \tau^{-1}\right)(r_0 - r_2).$$

Note that $\frac{d\beta_1}{d\alpha}\big|_{\alpha=0} \ge \frac{df}{d\alpha}\big|_{\alpha=0}$ if and only if (7) holds. Now, we can conclude that $\frac{d\beta_1}{d\alpha} \ge \frac{df}{d\alpha}$ for all $\alpha \ge 0$ because

$$\frac{d^2\beta_1}{d\alpha^2} = \frac{2}{\tau}\left[(\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha)^2 - 4\lambda(1+u\tau^{-1})\tau^{-1}\right]^{-3/2} > \frac{d^2 f}{d\alpha^2} = 0.$$

Hence, $\beta_1 \ge f$ for $\alpha \ge 0$, i.e., (EC.1) holds.

We next prove (EC.2). Since $\beta_1 = 1$ when $\alpha = 0$ and $d\beta_1/d\alpha < 0$, it is obvious that $\beta_1 < 1$ for any $\alpha > 0$. Furthermore, by the definition of β_2, we have $\beta_1 - \beta_2 = (1-\beta_1)u\tau^{-1} > 0$. To prove $\beta_2 \ge 0$, we need to show that $\beta_1 \ge u\tau^{-1}/(1+u\tau^{-1})$, which is equivalent to $(\lambda+\alpha)u \le 1$ after some algebraic manipulations. Since $\alpha < u^{-1} - \lambda$, we get $\beta_2 \ge 0$, which completes the proof of (EC.2).

To show (EC.3), we plug the expression of β_2 in (19) into (EC.3) and get $-\lambda(1+u\tau^{-1})\beta_1^2 + [\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha]\beta_1 - \tau^{-1} = 0$. It is straightforward to verify that β_1 defined in (18) is a solution to the above equation. Hence, (EC.3) holds.

Finally, taking the differences of the left-hand sides of (EC.3) and (EC.4), we get

$$\begin{aligned}&[\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1] - [\lambda\beta_2(1-\beta_2) - \tau^{-1}(1-\beta_2) + \alpha\beta_2]\\ &= \lambda(1-\beta_2)(\beta_1-\beta_2) + \tau^{-1}(\beta_1-\beta_2) + \alpha(\beta_1-\beta_2) \ge 0,\end{aligned}$$

which proves (EC.4). Similarly, taking the differences of the left-hand sides of (EC.3) and (EC.5), we get

$$\begin{aligned}&[\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1] - [\lambda\beta_2(1-\beta_2) - u^{-1}(\beta_1-\beta_2) + \alpha\beta_2]\\ &= \lambda(1-\beta_2)(\beta_1-\beta_2) + \alpha(\beta_1-\beta_2) \ge 0,\end{aligned}$$

which proves (EC.5). □

Proof of Lemma 4. We first show part (i), i.e., h1) through h11) are preserved under L. We first show that L preserves h1) by considering three separate cases. For $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \ge 2$ and $x_2 \ge 0$, we have

$$LG(\boldsymbol{x}) = \lambda G(\boldsymbol{x} + \boldsymbol{e_1}) + u^{-1}\max\{G(\boldsymbol{x}), 0\} + \tau^{-1}\min\{G(\boldsymbol{x}), 0\} + q_2u^{-1}\min\{G(\boldsymbol{x} - \boldsymbol{e_1} + \boldsymbol{e_3}), 0\}$$

$$+\tau^{-1}\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}+q_1(\tau u)^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})]+r_0\tau^{-1}. \quad \text{(EC.6)}$$

Then, by h1) and h2), each term of $LG(\boldsymbol{x})$ is less than or equal to the corresponding term of $LG(\boldsymbol{x}+\boldsymbol{e_3})$ thus $LG(\boldsymbol{x})\le LG(\boldsymbol{x}+\boldsymbol{e_3})$. When $\boldsymbol{x}=(1,0,x_2)$ with $x_2\ge 1$, we have

$$LG(\boldsymbol{x})=\lambda G(\boldsymbol{x}+\boldsymbol{e_1})+u^{-1}G(\boldsymbol{x})+\tau^{-2}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]+r_0\tau^{-1}. \quad \text{(EC.7)}$$

Then, by h1) and h3), $LG(\boldsymbol{x})\le LG(\boldsymbol{x}+\boldsymbol{e_3})$ for $\boldsymbol{x}=(1,0,x_2)$ with $x_2\ge 1$. Finally, we have

$$LG(\boldsymbol{e_1})=\lambda G(2\boldsymbol{e_1})+u^{-1}G(\boldsymbol{e_1})+r_0\tau^{-1}. \quad \text{(EC.8)}$$

Then, by h1), $LG(\boldsymbol{e_1})\le LG(\boldsymbol{e_1}+\boldsymbol{e_3})$, which completes the proof that h1) is preserved under L.

Now we show that h2) is preserved under L. For $x_0\ge 0,\ x_1=0$, and $x_2\ge 0$, we have

$$\begin{aligned}&Lv(\boldsymbol{x}+\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_2})\\&=\lambda[v(\boldsymbol{x}+2\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})]+\min\{G(\boldsymbol{x}+\boldsymbol{e_1}),0\}+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_2})]+r_0-r_1.\end{aligned}$$

By h1) and h2), we have $Lv(\boldsymbol{x}+\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_2})\le Lv(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_2}+\boldsymbol{e_3})$, which completes the proof that h2) is preserved under L.

Next, we show that h3) is preserved by considering three separate cases. For $\boldsymbol{x}=\boldsymbol{0}$, we have

$$\begin{aligned}&Lv(\boldsymbol{e_3})-Lv(\boldsymbol{0})=\lambda[v(\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{e_1})]+u^{-1}[v(\boldsymbol{e_3})-v(\boldsymbol{0})]+r_2,\\&Lv(2\boldsymbol{e_3})-Lv(\boldsymbol{e_3})=\lambda[v(\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{e_1}+\boldsymbol{e_3})]+\tau^{-1}[v(\boldsymbol{e_3})-v(\boldsymbol{0})]+u^{-1}[v(2\boldsymbol{e_3})-v(\boldsymbol{e_3})]+r_2.\end{aligned}$$

Then, by h3), we obtain $Lv(\boldsymbol{e_3})-Lv(\boldsymbol{0})\le Lv(2\boldsymbol{e_3})-Lv(\boldsymbol{e_3})$. For $\boldsymbol{x}=(0,0,x_2)$ with $x_2\ge 1$, we have

$$Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+r_2.$$

Then, by h3), we have $Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})\le Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})$. For $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\ge 1$ and $x_2\ge 0$, we have

$$\begin{aligned}Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})&=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1})]\\&\quad+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+\min\{G(\boldsymbol{x}+\boldsymbol{e_3}),0\}-\min\{G(\boldsymbol{x}),0\}+r_2.\end{aligned}$$

To compare $Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$ and $Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})$ for $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\ge 1$ and $x_2\ge 0$, we look at three separate cases: (i) If $G(\boldsymbol{x})\ge 0$, then by h1), $G(\boldsymbol{x}+2\boldsymbol{e_3})\ge G(\boldsymbol{x}+\boldsymbol{e_3})\ge 0$, and hence $Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})\ge Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$ by h3); (ii) If $G(\boldsymbol{x})<0$ and $G(\boldsymbol{x}+2\boldsymbol{e_3})\ge 0$, we have

$$
\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})\\
&\le \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+u^{-1}[q_1 v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})+q_2 v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})]+\tau^{-1}v(\boldsymbol{x}+\boldsymbol{e_3})+r_2\\
&\quad -u^{-1}[q_1 v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})+q_2 v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]-\tau^{-1}v(\boldsymbol{x})\\
&=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+q_1u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]+\tau^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]\\
&\quad +q_2u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+r_2, \text{ and}\\
&Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})\\
&\ge \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})]+\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3})]+r_2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&Lv(\boldsymbol{x}+2\boldsymbol{e_3})-2Lv(\boldsymbol{x}+\boldsymbol{e_3})+Lv(\boldsymbol{x})\\
&\ge \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]\\
&\quad +\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3})]-\tau^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]\\
&\quad -q_1u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]-q_2u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&\ge q_1u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]\\
&\quad +q_1u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))]\\
&\quad +q_2u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad -\tau^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\ge (q_2u^{-1}-\tau^{-1})[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]>0,
\end{aligned}
$$

where the second inequality holds by h3), the third inequality holds by h2) and h3), and the last inequality holds by h4) and the condition $u<\tilde{u}(\alpha)$ and the fact $\tilde{u}(\alpha)<q_2\tau$. (iii) If $G(\boldsymbol{x}+2\boldsymbol{e_3})<0$, then by h1), we have $G(\boldsymbol{x})\le G(\boldsymbol{x}+\boldsymbol{e_1})<0$. Then, we obtain

$$
\begin{aligned}
&Lv(\boldsymbol{x}+2\boldsymbol{e_3})-2Lv(\boldsymbol{x}+\boldsymbol{e_3})+Lv(\boldsymbol{x})\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]\\
&\quad +q_1u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))]\\
&\quad +q_2u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+3\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad +\tau^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]\ge 0,
\end{aligned}
$$

where the inequality holds by h3). This completes the proof that h3) is preserved under L.

We next show that h4) is preserved by considering two separate cases. For $\boldsymbol{x}=(1,0,x_2)$ with $x_2\ge 0$, by h1) and h4), we have

$$
\begin{aligned}
&[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]-[Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]
\end{aligned}
$$

$$+u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\geq 0.$$

For $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\geq 2$ and $x_2\geq 0$, we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]-[Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))\\
&\quad-(v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))]+u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad+\min\{G(\boldsymbol{x}+\boldsymbol{e_3}),0\}+\min\{G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}),0\}-\min\{G(\boldsymbol{x}),0\}-\min\{G(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}),0\}. \qquad \text{(EC.9)}
\end{aligned}$$

To compare $Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$ and $Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})$, we look at three separate cases. (i) When $G(\boldsymbol{x}+\boldsymbol{e_3})\geq 0$ and $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})\geq 0$, by h1) we have $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{2e_3})\geq 0$. Hence, by (EC.9)

$$\begin{aligned}
&[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]-[Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&\geq\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))-(v(\boldsymbol{x}-2\boldsymbol{e_1}\\
&\quad+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))]+u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\geq 0,
\end{aligned}$$

where the last inequality holds because of h4).

(ii) When $G(\boldsymbol{x}+\boldsymbol{e_3})<0$ and $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})\geq 0$, by h1) we have $G(\boldsymbol{x})<0$ and $G(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})>0$. Hence, by (EC.9), we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]-[Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]\\
&\quad+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))-(v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad+u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]+G(\boldsymbol{x}+\boldsymbol{e_3})-G(\boldsymbol{x})\geq 0,
\end{aligned}$$

where the last inequality holds because of h1) and h4).

(iii) Otherwise, i.e., $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})<0$, we have $G(\boldsymbol{x}+\boldsymbol{e_3})<0$ by h5). Hence, by (EC.9), we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]-[Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&\geq\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))\\
&\quad-(v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))]+u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad+G(\boldsymbol{x}+\boldsymbol{e_3})+G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-G(\boldsymbol{x})-G(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]\\
&\quad+\tau^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad+q_1u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))-(v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}))]\\
&\quad+q_2u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-2\boldsymbol{e_1}+3\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3}))]\geq 0,
\end{aligned}$$

where the last inequality follows from h4). This completes the proof that h4) is preserved under L.

We next show that h5) is preserved by considering three separate cases. When $x_0 \geq 2,\ x_1 = 0$, and $x_2 \geq 0$, then $LG(\boldsymbol{x})$ satisfies (EC.6), which also gives

$$LG(\boldsymbol{x}+\boldsymbol{e_1}) = \lambda G(\boldsymbol{x}+2\boldsymbol{e_1}) + u^{-1}G(\boldsymbol{x}+\boldsymbol{e_1}) + r_0\tau^{-1} + \tau^{-1}G(\boldsymbol{x}) + q_1(\tau u)^{-1}[v(\boldsymbol{x}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]. \quad \text{(EC.10)}$$

Then, by h5) and h6), $LG(\boldsymbol{x}) \geq LG(\boldsymbol{x}+\boldsymbol{e_1})$. When $\boldsymbol{x} = (1, 0, x_2)$ with $x_2 \geq 1$, then $LG(\boldsymbol{x})$ satisfies (EC.7) and $LG(\boldsymbol{x}+\boldsymbol{e_1})$ is given by (EC.10). By h5), h7) and h8) we have

$$LG(\boldsymbol{x}+\boldsymbol{e_1}) - LG(\boldsymbol{x}) \leq \tau^{-1}G(\boldsymbol{x}) - q_1(\tau u)^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x})] - \tau^{-2}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] \leq 0.$$

When $\boldsymbol{x} = \boldsymbol{e_1}$, $LG(\boldsymbol{e_1})$ is given by (EC.8) and $LG(2\boldsymbol{e_1})$ is given by (EC.6). By h5), h7), and h10), we have

$$LG(2\boldsymbol{e_1}) - LG(\boldsymbol{e_1}) \leq \tau^{-1}G(\boldsymbol{e_1}) + q_1(\tau u)^{-1}(v(\boldsymbol{e_1}) - v(\boldsymbol{e_2})) \leq \tau^{-1}r_0(1 - \tilde{u}(\alpha)/u) < 0,$$

where we use the condition that $u < \tilde{u}(\alpha)$.

Next, we show that h6) is preserved. For any $\boldsymbol{x} \in \mathcal{S}$ with $x_1 = 0$, we have

$$\begin{aligned} Lv(\boldsymbol{x}+\boldsymbol{e_1}) - Lv(\boldsymbol{x}+\boldsymbol{e_2}) &= \lambda[v(\boldsymbol{x}+2\boldsymbol{e_1}) - v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})] + \min\{G(\boldsymbol{x}+\boldsymbol{e_1}), 0\} \\ &\quad + u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_1}) - v(\boldsymbol{x}+\boldsymbol{e_2})] + r_0 - r_1, \text{ and} \\ Lv(\boldsymbol{x}+2\boldsymbol{e_1}) - Lv(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) &= \lambda[v(\boldsymbol{x}+3\boldsymbol{e_1}) - v(\boldsymbol{x}+2\boldsymbol{e_1}+\boldsymbol{e_2})] + \min\{G(\boldsymbol{x}+2\boldsymbol{e_1}), 0\} \\ &\quad + u^{-1}[v(\boldsymbol{x}+2\boldsymbol{e_1}) - v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})] + r_0 - r_1. \end{aligned}$$

Then, by h5) and h6), it is obvious that $Lv(\boldsymbol{x}+\boldsymbol{e_1}) - Lv(\boldsymbol{x}+\boldsymbol{e_2}) \geq Lv(\boldsymbol{x}+2\boldsymbol{e_1}) - Lv(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})$.

Now, we show that h7) is preserved. For $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \geq 0$ and $x_2 \geq 0$, we have

$$\begin{aligned} q_1[Lv(\boldsymbol{x}+\boldsymbol{e_2}) - Lv(\boldsymbol{x}+\boldsymbol{e_1})] &\geq q_1[\lambda(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}+2\boldsymbol{e_1})) + u^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2}) - v(\boldsymbol{x}+\boldsymbol{e_1})) + r_1 - r_0] \\ &\geq (\lambda + u^{-1})\tilde{u}(\alpha)r_0 + q_1(r_1 - r_0) = (\lambda + u^{-1})\tilde{u}(\alpha)r_0 + q_2(r_0 - r_2) = \tilde{u}(\alpha)r_0, \end{aligned}$$

where the inequality holds by h7).

Next, we show that h8) is preserved by considering three separate cases. When $\boldsymbol{x} = \boldsymbol{e_1} + \boldsymbol{e_3}$, by (EC.7) we have

$$\begin{aligned} &LG(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}[Lv(\boldsymbol{e_3}) - Lv(\boldsymbol{0})] \\ &\leq \lambda[G(2\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{e_1}))] + u^{-1}[G(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0}))] + \tau^{-2}[v(\boldsymbol{e_3}) - v(\boldsymbol{0})] \\ &\quad + (r_0 - r_2)\tau^{-1} \\ &\leq (\lambda + u^{-1})\tilde{u}(\alpha)r_0u^{-1} + u^{-1}[G(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0})) - \tilde{u}(\alpha)r_0u^{-1}] + q_2u^{-1}(r_0 - r_2) \\ &= \tilde{u}(\alpha)r_0u^{-1} + u^{-1}[G(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0})) - \tilde{u}(\alpha)r_0u^{-1}] \leq \tilde{u}(\alpha)r_0u^{-1}, \end{aligned}$$

where h8), h11), and (EC.1) are used in the second inequality, and the last inequality follows by h8) if $G(\boldsymbol{e_1}+\boldsymbol{e_3}) > 0$.

When $\boldsymbol{x}=(1,0,x_2)$ with $x_2\geq 2$, by (EC.7) we have

$$
\begin{aligned}
&LG(\boldsymbol{x})-\tau^{-1}[Lv(\boldsymbol{x}-\boldsymbol{e_1})-Lv(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]\\
&\leq \lambda[G(\boldsymbol{x}+\boldsymbol{e_1})-\tau^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3})]]+u^{-1}[G(\boldsymbol{x})-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]]\\
&\quad+\tau^{-2}[(v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}-2\boldsymbol{e_3}))]+\tau^{-1}(r_0-r_2)\\
&\leq (\lambda+u^{-1})\tilde{u}(\alpha)r_0u^{-1}+u^{-1}[G(\boldsymbol{x})-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]-\tilde{u}(\alpha)r_0u^{-1}]+q_2u^{-1}(r_0-r_2)\\
&=\tilde{u}(\alpha)r_0u^{-1}+u^{-1}[G(\boldsymbol{x})-\tau^{-1}(v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}))-\tilde{u}(\alpha)r_0u^{-1}]\leq \tilde{u}(\alpha)r_0u^{-1},
\end{aligned}
$$

where the first inequality follows by h8) and h9), and the last inequality follows by h8) if $G(\boldsymbol{x})>0$. When $x_0\geq 2$, $x_1=0$, and $x_2\geq 1$, by (EC.6) we have

$$
\begin{aligned}
&LG(\boldsymbol{x})-\tau^{-1}(Lv(\boldsymbol{x}-\boldsymbol{e_1})-Lv(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}))\\
&=\lambda[G(\boldsymbol{x}+\boldsymbol{e_1})-\tau^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3})]]+u^{-1}[\max\{G(\boldsymbol{x}),0\}-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]]\\
&\quad+\tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}-\tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})]]-q_1(\tau u)^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})-v(\boldsymbol{x}-\boldsymbol{e_1})]\\
&\quad+\tau^{-1}(\min\{G(\boldsymbol{x}),0\}-\min\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\})+q_2u^{-1}\min\{G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}),0\}\\
&\quad+\tau^{-1}\min\{G(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}),0\}+(r_0-r_2)\tau^{-1}\\
&\leq \lambda[G(\boldsymbol{x}+\boldsymbol{e_1})-\tau^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3})]]+u^{-1}[\max\{G(\boldsymbol{x}),0\}-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]]\\
&\quad+\tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}-\tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})]]-\tilde{u}(\alpha)r_0(\tau u)^{-1}+(r_0-r_2)\tau^{-1},
\end{aligned}
$$

where the last inequality holds because of h5) and h7). Then, by h8) we have

$$
\begin{aligned}
&LG(\boldsymbol{x})-\tau^{-1}(Lv(\boldsymbol{x}-\boldsymbol{e_1})-Lv(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}))\\
&\leq (\lambda+u^{-1})\tilde{u}(\alpha)r_0u^{-1}+u^{-1}[\max\{G(\boldsymbol{x}),0\}-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]-\tilde{u}(\alpha)r_0u^{-1}]\\
&\quad+\tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}-\tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})]-\tilde{u}(\alpha)r_0u^{-1}]+(r_0-r_2)\tau^{-1}\\
&=\tilde{u}(\alpha)r_0u^{-1}+u^{-1}[\max\{G(\boldsymbol{x}),0\}-\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]-\tilde{u}(\alpha)r_0u^{-1}]+(r_0-r_2)\tau^{-1}\\
&\quad+\tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}-\tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})]-\tilde{u}(\alpha)r_0u^{-1}]-(\alpha+\tau^{-1})\tilde{u}(\alpha)r_0u^{-1}\\
&\leq \tilde{u}(\alpha)r_0u^{-1}+(r_0-r_2)(\tau^{-1}-q_2u^{-1})<\tilde{u}(\alpha)r_0u^{-1},
\end{aligned}
$$

where the last inequality holds because $r_0>r_2$ and $u<\tilde{u}(\alpha)<q_2\tau$.

Now, we show that h9) is preserved under L by considering four separate cases. When $\boldsymbol{x}\in\mathcal{S}$ and $x_1\geq 1$, by h9) we have

$$
\begin{aligned}
&[Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})]-[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_2}+2\boldsymbol{e_3})-v(\boldsymbol{x}\\
&\quad-\boldsymbol{e_2}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_2}))]+u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]\\
&\leq (1-\alpha)(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2)<(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2).
\end{aligned}
$$

When $\boldsymbol{x}=\boldsymbol{0}$, by h9), h11), and (EC.1), we have

$$[Lv(2\boldsymbol{e_3})-Lv(\boldsymbol{e_3})]-[Lv(\boldsymbol{e_3})-Lv(\boldsymbol{0})]$$
$$=\lambda[(v(\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{e_1}))]+\tau^{-1}(v(\boldsymbol{e_3})-v(\boldsymbol{0}))$$
$$+u^{-1}[(v(2\boldsymbol{e_3})-v(\boldsymbol{e_3}))-(v(\boldsymbol{e_3})-v(\boldsymbol{0}))]\leq(1-\alpha)(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2)<(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2).$$

When $\boldsymbol{x}=(0,0,x_2)$ with $x_2\geq 1$, by h9) we have

$$[Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})]-[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]$$
$$=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]+\tau^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))$$
$$-(v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3}))]+u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]$$
$$\leq(1-\alpha)(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2)<(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2).$$

Finally, when $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\geq 1$ and $x_2\geq 0$, we consider two subcases. If $G(\boldsymbol{x}+\boldsymbol{e_3})\geq 0$, by h9) we have

$$[Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})]-[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]$$
$$\leq\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))$$
$$-(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))]+u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]$$
$$\leq(1-\alpha)(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2)<(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2).$$

If $G(\boldsymbol{x}+\boldsymbol{e_3})<0$, by h9) we have

$$[Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})]-[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]$$
$$\leq\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]$$
$$+q_1u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))]$$
$$+q_2u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+3\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]$$
$$+\tau^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]$$
$$\leq(1-\alpha)(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2)<(q_2u^{-1}-\tau^{-1})\tau^2(r_0-r_2).$$

Now, we show that h10) is preserved by considering two separate cases. When $\boldsymbol{x}=(x_0,0,0)$ with $x_0\geq 2$, from (EC.6) we have

$$LG(\boldsymbol{x})$$
$$\leq\lambda G(\boldsymbol{x}+\boldsymbol{e_1})+u^{-1}\max\{G(\boldsymbol{x}),0\}+\tau^{-1}\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}+q_1(\tau u)^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})]+r_0\tau^{-1}$$
$$\leq(\lambda+u^{-1}+\tau^{-1})r_0+\tau^{-1}(r_0-q_1u^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})-v(\boldsymbol{x}-\boldsymbol{e_1})])\leq(1-\alpha)r_0<r_0,$$

where the second inequality holds by h10) and the third inequality holds by h7) and the condition that $u<\tilde{u}(\alpha)$. When $\boldsymbol{x}=\boldsymbol{e_1}$, from (EC.8) we have

$$LG(\boldsymbol{e_1})\leq\lambda G(2\boldsymbol{e_1})+u^{-1}\max\{G(\boldsymbol{e_1}),0\}+r_0\tau^{-1}\leq(\lambda+u^{-1}+\tau^{-1})r_0=(1-\alpha)r_0<r_0,$$

where the second inequality holds by h10).

Next, we show that h11) is preserved under L by considering four cases. When $x_0 \geq 0,\ x_1 \geq 1$, and $x_2 \geq 0$, we have

$$\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})\\
&= \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})] + \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_2}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_2})] + u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})] + r_2\\
&\leq \lambda r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0+1}] + \tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_1+x_2}\beta_2^{x_0}] + u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}] + r_2\\
&= r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}] + r_2\alpha^{-1}\beta_1^{x_1+x_2}\beta_2^{x_0}[\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1] = r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}],
\end{aligned}$$

where the inequality holds by h11) and (EC.3). When $\boldsymbol{x} = \boldsymbol{0}$, by h11) we have

$$\begin{aligned}
Lv(\boldsymbol{e_3}) - Lv(\boldsymbol{0}) &= \lambda[v(\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{e_1})] + u^{-1}[v(\boldsymbol{e_3}) - v(\boldsymbol{0})] + r_2\lambda r_2\alpha^{-1}(1-\beta_1\beta_2) + u^{-1}r_2\alpha^{-1}(1-\beta_1) + r_2\\
&= r_2\alpha^{-1}(1-\beta_1) + r_2\alpha^{-1}[\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1] = r_2\alpha^{-1}(1-\beta_1),
\end{aligned}$$

where the last equality holds by (EC.3). When $\boldsymbol{x} = (0, 0, x_2)$ with $x_2 \geq 1$, by h11) we have

$$\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x}) = \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})] + \tau^{-1}[v(\boldsymbol{x}) - v(\boldsymbol{x}-\boldsymbol{e_3})] + u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})] + r_2\\
&\leq \lambda r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2] + \tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2}] + u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}] + r_2\\
&= r_2\alpha^{-1}[1-\beta_1^{x_2+1}] + r_2\alpha^{-1}\beta_1^{x_2}[\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1] = r_2\alpha^{-1}(1-\beta_1^{x_2+1}),
\end{aligned}$$

where the last equality holds by (EC.3). When $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \geq 1$ and $x_2 \geq 0$, we have

$$\begin{aligned}
Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x}) &= \lambda v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) + \min\{G(\boldsymbol{x}+\boldsymbol{e_3}), 0\} + \tau^{-1}v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) + u^{-1}v(\boldsymbol{x}+\boldsymbol{e_3})\\
&\quad + r_2 - \lambda v(\boldsymbol{x}+\boldsymbol{e_1}) - \min\{G(\boldsymbol{x}), 0\} - \tau^{-1}v(\boldsymbol{x}-\boldsymbol{e_1}) - u^{-1}v(\boldsymbol{x}).
\end{aligned} \tag{EC.11}$$

We consider two separate cases. If $G(\boldsymbol{x}) \geq 0$, then from (EC.11), we have

$$\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})\\
&\leq \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})] + \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1})] + u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})] + r_2\\
&\leq \lambda r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0+1}] + \tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0-1}] + u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}] + r_2\\
&= r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}] + r_2\alpha^{-1}\beta_1^{x_2+1}\beta_2^{x_0-1}[\lambda\beta_2(1-\beta_2) - \tau^{-1}(1-\beta_2) + \alpha\beta_2] \leq r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}],
\end{aligned}$$

where the second inequality holds by h11) and the last inequality holds by (EC.2) and (EC.4). If $G(\boldsymbol{x}) < 0$, then from (EC.11), we have

$$\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x}) \leq \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})] + q_1u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]\\
&\quad + q_2u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})] + \tau^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})] + r_2\\
&\leq \lambda r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0+1}] + q_1u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+2}\beta_2^{x_0-1}] + q_2u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+2}\beta_2^{x_0-1}]\\
&\quad + \tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}] + r_2\\
&= r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}] + r_2\alpha^{-1}\beta_1^{x_2+1}\beta_2^{x_0-1}[\lambda\beta_2(1-\beta_2) - u^{-1}(\beta_1-\beta_2) + \alpha\beta_2] \leq r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}],
\end{aligned}$$

where the second inequality holds by h11) and the last inequality holds by (EC.2) and (EC.5).

We prove part (ii) by verifying the conditions in Theorem 6.11.3 of Puterman (2005). It is obvious that the state space $\mathcal{S}$ is countable and we have already verified Assumptions 6.10.1 and 6.10.2 of Puterman (2005) in our proof of Proposition 4. Hence, we only need to show that conditions (a), (b), and (c) in Theorem 6.11.3 of Puterman (2005) hold. Condition (a) holds by part (i). Next, consider a stationary policy π that serves a class 1 customer if $x_1 \geq 1$ and serves a class 2 customer only if $x_0 = x_1 = 0$. Furthermore, under π, if the server triages a class 0 customer in $(x_0, 0, x_2)$ for $x_0 \geq 1$ and $x_2 \geq 0$, then the server will perform triage in $(x_0', 0, x_2')$ where $x_0' \geq x_0$ and $0 \leq x_2' \leq x_2$. From the optimality equations, we find that $v \in \mathbb{H}$ implies that policy π is an optimal policy. Hence, condition (b) holds. Finally, condition (c) holds, i.e., $\mathbb{H}$ is closed, because the limit of any convergent sequence of functions that satisfy h1) through h11) will satisfy them as well. Hence, there exists an optimal stationary policy whose value function belongs to $\mathbb{H}$. $\square$